ECE560: Computer Systems Performance Evaluation

Lecture 14 (Reference)

Embedded Markov-Chain Queueing Systems - Derivations

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Topics

- Transform Methods
- M/G/1, M/D/1
- GI/M/1

Solution:

- Constructing an embedded Markov chain
- And applying *z*-transform and Laplace-Stieltjes transform methods

"Embedded Markov Chain Queueing Systems"

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Transform Methods (Review)

- Moment generating function (MGF)
- Probability generating function (PGF, z-transform)
- Laplace-Stieltjes transform (LST)

Chapter 2.9 in Allen's book

Preparation for Embedded Markov chain queueing systems

Transform Methods

- Goal

  To transform a *r.v.* into some transformed function with a different domain, in which it is easier to perform operations such as finding the mean, the variance, the moments.

Transform Methods

- Example: Logarithm
  - One of the first transform methods used successfully
  - Transform the problem of multiplying 2 large numbers A and B into the simpler problem of adding 2 numbers log$A$ and log$B$.

$$\log(A \times B) = \log A + \log B$$

  - To complete operation, calculating "anti-logarithm"

$$A \times B = e^{\log A + \log B}$$

Multiplication → Addition

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Agenda (Transform Methods)

- **Moment generating function (MGF)**
- Probability generating function (PGF, z-transform)
- Laplace-Stieltjes transform (LST)

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Moment Generating Function (MGF) (1)

- **Definition**: the MGF of a *r.v.* X is defined by $\psi_X[\theta] = E[e^{\theta X}]$ for all real θ such that $E[e^{\theta X}]$ is finite. Thus,

$$\Psi_X[\theta] = \begin{cases} \sum_{x_i} e^{\theta x_i} p(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{\theta x} f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

- Notes:
  - $\Psi_X[0] = 1$
  - A r.v. X has a MGF *iff* all the moments of X exist / are finite
  - MGF transforms the r.v. X defined on a sample space into the function $\Psi_X[\bullet]$ defined on some set of real #s

Moment Generating Function (MGF) (2)

- Properties (*Theorem 2.9.1*)
  - Uniqueness: X and Y have the same distribution ($F_X = F_Y$) *iff* $\Psi_X[\bullet] = \Psi_Y[\bullet]$
  - Moment generating property:
    - The *n*th moment of X :

$$E[X^n] = \left.\frac{d^n \Psi_X[\theta]}{d\theta^n}\right|_{\theta=0} = \Psi_X^{(n)}[0]$$

    - Hence,

$$E[X] = \Psi_X{}'[0]$$

$$E[X^2] = \Psi_X{}''[0]$$

$$\sigma^2 = E[X^2] - E[X]^2 = \Psi_X{}''[0] - (\Psi_X{}'[0])^2$$

  - Convolution property:
    - If X ∐ Y,

$$\Psi_{X+Y}[\theta] = \Psi_X[\theta] \cdot \Psi_Y[\theta]$$

Moment Generating Function (MGF) (3) – Hands-on Problem

- Let X be a Poisson r.v. with rate λ, Find E[X] and Var[X] using its MGF.

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Agenda (Transform Methods)

- Moment generating function (MGF)
- **Probability generating function (PGF, z-transform)**
- Laplace-Stieltjes transform (LST)

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Probability Generating Function (PGF, *z*-transform) (1)

- **Definition**: Given a non-negative integer-valued *discrete* r.v. X with *p.m.f* of $\boldsymbol{p[X=k]=p[k]=p_k}$, define the PGF of X by

$$g_X[z]=E[z^X]=\sum_{i=0}^{\infty}p_i z^i = p_0+p_1 z+p_2 z^2+\cdots$$

  - Note:

$$g_X[1]=1=\sum_{i=0}^{\infty}p_i$$

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z-transform (2)

- Properties (***Theorem 2.9.2***)
  - Uniqueness
    - r.v. X and Y have the same distribution ($F_X=F_Y$) *iff* $g_X[z]=g_Y[z]$
  - Moment generating property

$$p_n=p[X=n]=\frac{1}{n!}\frac{d^n g_X[z]}{dz^n}\bigg|_{z=0}=\frac{1}{n!}g_X^{(n)}[0]$$

$$n=0,1,2\cdots$$

$$E[X]=g'_X[1]$$

$$Var[X]=g''_X[1]+g'_X[1]-(g'_X[1])^2$$

  - Convolution property

$$\boldsymbol{g_{X+Y}[z]=g_X[z]\cdot g_Y[z] \quad \textbf{if} \quad X \coprod Y}$$

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z-transform (3) – Hands-on Problem

- Let X be a Bernoulli *r.v.* which describes a Bernoulli trial. Find E[X] and Var[X] using its PGF/z-transform.

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Agenda (Transform Methods)

- Moment generating function (MGF)
- Probability generating function (PGF, z-transform)
- **Laplace-Stieltjes transform (LST)**

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Laplace-Stieltjes Transform (LST) (1)

- **Definition**:
  - let X be a r.v. such that $p[X<0]=0$ (non-negative) then the LST of X is defined as:

$$X^*(\theta)=L_X[\theta]=\Psi_X[-\theta]=E[e^{-\theta X}]$$

$$\begin{cases} \sum_{x_i} e^{-\theta x_i} p(x_i) & \text{if } X \text{ is discrete} \\ \int_0^{\infty} e^{-\theta x} f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

- Notes:
  - For continuous X, the integral $\int_0^{\infty} e^{-\theta x} f(x)dx$ is called the Laplace transform of the function $f(x)$
  - $\int_0^{\infty} e^{-\theta x} f(x)dx$ is also written as $\int_0^{\infty} e^{-\theta x} dF(x)$ which is called a "stielgjes integral"

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Laplace-Stieltjes Transform (LST) (2)

- Properties (***Theorem 2.9.3***)
  - Uniqueness
    - r.v. X and Y have the same distribution ($F_X = F_Y$) iff $L_X[z]=L_Y[z]$
  - Moment generating property: For $\theta > 0$,
    - $L_X[\theta]=X^*$ has derivatives of all orders given by:

$$\frac{d^n X^*}{d\theta^n} = \begin{cases} (-1)^n \int_0^{\infty} e^{-\theta x} x^n f(x)dx & \text{X is continuous} \\ (-1)^n \sum_{x_i} e^{-\theta x_i} x_i^n p(x_i) & \textbf{X is discrete} \end{cases}$$

- If $E[X^n]$ exists, then,

$$E[X^n]=(-1)^n \left.\frac{d^n X^*[\theta]}{d\theta^n}\right|_{\theta=0} = (-1)^n X^{*(n)}(0)$$

$$\boldsymbol{E[X]=-L'_X[0],\ E[X^2]=L''_X[0]}$$

  - Convolution property
    - **if** $\boldsymbol{X \coprod Y}$,

$$(X+Y)^*[\theta] = X^*[\theta]Y^*[\theta]$$

or $\boldsymbol{L_{X+Y}[\theta]=L_X[\theta]\cdot L_Y[\theta]}$

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LST (3) – Hands-on Problem

- Let X be an exponential r.v. with parameter λ. Find E[X] and Var[X] using its LST.

Summary of Transform Methods

- Useful transform methods: to transform a *r.v.* into some transformed function with a different domain, in which it is easier to perform operations such as finding the mean, the variance, the moments.
  - Moment generating function (MGF)

    $\psi_X[\theta] = E[e^{\theta X}]$ and $E[X^n] = \psi_X^{(n)}[0]$

  - Probability generating function (PGF/z-transform/Generating function)
    - Non-negative integer-valued discrete r.v.s

      $g_X[z] = E[z^X]$ and $E[X] = g'_X[1]$,
      $Var[X] = g''_X[1] + g'_X[1] - (g'_X[1])^2$

  - Laplace-Stieltjes transform (LST)
    - Non-negative r.v.s

      $X^*[\theta] = \psi_X[-\theta] = E[e^{-\theta X}]$ and
      $E[X^n] = (-1)^n X^{*(n)}[0]$

*** unique**
*** MGP**
*** convolution.**

Topics

- Transform Methods
- **<u>M/G/1, M/D/1</u>**
- GI/M/1

Solution:
- Constructing an embedded Markov chain
- And applying *z*-transform and Laplace-Stieltjes transform methods

"Embedded Markov Chain Queueing Systems"

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M/G/1 Queueing Systems

- Assume
  - Poisson arrival process with rate λ
  - General service time distribution with
    - different customers have independent service times
    - $E[s]$ and $E[s^2]$ exist (in order to calculate L,W)

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M/G/1 (Cont'd)

- $\{N(t),\ t \geq 0\}$ representing the number of customers in the system at time t is NOT a Markov process

Service time is not exponential! Future value of N(t) depends not only on the current value of N(t), but also on the remaining service time for the current customer!

- However, by explaining $N(t)$ only at instants of departure, we can define an embedded Markov chain
  - Let $0<t_1<t_2<\ldots<t_n<\ldots$ denote the successive times at which a customer completes service
  - $X_n = N(t_n)$: number of customers the n^{th} departing customer leaves behind
  - $\{ X_n \}$ is a Markov chain

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M/G/1 (Cont'd)

- Explain that $\{ X_n \}$ is a Markov chain
  - Let A denote the number of customers who arrive for service during "service time of $(n+1)$st customer" (denoted by s)

$$X_{n+1} = \begin{cases} X_n - 1 + A & \text{if } X_n \geq 1 \\ A & \text{if } X_n = 0 \end{cases}$$

  - s is independent of service times of other customers and of the number of customers in the system
  - The arrival process is Poisson, which has stationary increments → A depends only on s and not on when the service began
  - X_{n+1} depends only on the value of X_n, and independent r.v. A, not on X_{n-1}, X_{n-2}, …

Therefore, $\{ X_n \}$ is a Markov chain

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One-step transition matrix P of the embedded Markov chain $\{X_n\}$?

Find Transition Matrix P of $\{X_n\}$

- Since arrivals are Poisson

  $P[A=n|s=t] = e^{-\lambda t}(\lambda t)^n / n!, \ n=0, 1, 2, \ldots$

- Therefore

$$P_{ij} = P[X_{n+1} = j \mid X_n = i]$$
$$= P[A = j-i+1]$$
$$= \int_0^\infty P[A = j-i+1 \mid s = t]\, dW_s(t)$$

by "Total Probability Law"

$W_s(.)$: c.d.f. of the service time s

$$= \begin{cases} \int_0^\infty e^{-\lambda t} \dfrac{(\lambda t)^{j-i+1}}{(j-i+1)!} dW_s(t), & j-i+1 \ge 0, i \ge 1 \\ 0, & j-i+1 < 0\ (j < i-1), i \ge 1 \end{cases}$$

Why? departing customers can't leave behind fewer than "one less than that are found in the present state: i-1"

Find P of $\{X_n\}$(Cont'd)

- Let $k_n =$ Pr[n customers arrive during one service interval]

$$k_n = P[A=n] = \int_0^\infty P[A=n \mid s=t]\, dW_s(t)$$
$$= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dW_s(t), \quad n = 0,1,2,\ldots$$

- Then:

$$P_{ij} = \begin{cases} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j-i+1}}{(j-i+1)!} dW_s(t) = k_{j-i+1}, & j \geq i-1, i \geq 1 \\ 0, & j < i-1, i \geq 1 \end{cases}$$

- We have:

$$P = [P_{ij}] = \begin{bmatrix} ? & ? & ? & ? & ? & ? & \ldots \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \ldots \\ 0 & k_0 & k_1 & k_2 & k_3 & k_4 & \ldots \\ 0 & 0 & k_0 & k_1 & k_2 & k_3 & \ldots \\ 0 & 0 & 0 & k_0 & k_1 & k_2 & \ldots \\ 0 & 0 & 0 & 0 & k_0 & k_1 & \ldots \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

- How do we find the first row of P?
  - If a departing customer leaves NO customers behind ($X_n=i=0$), then no departure can occur until a new customer Z arrives.
  - The number left behind by that customer Z is simply the number that arrive during his service interval.

*n*th customer left behind: *i*	New arrival	Arrivals during service of (n+1)st customer	(n+1)st customer left behind: *j*
0	→ 1	→ k	→ k
1		→ k	→ k
$i \geq 1$		→ k	→ i-1+k

  - Therefore, the state transition probabilities are the same for $i=0$ as for $i=1$, i.e., the first row of P = the second row of P: $P_{0k}=P_{1k}=Pr(A=k)$

$$P = [P_{ij}] = \begin{bmatrix} k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \ldots \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \ldots \\ 0 & k_0 & k_1 & k_2 & k_3 & k_4 & \ldots \\ 0 & 0 & k_0 & k_1 & k_2 & k_3 & \ldots \\ 0 & 0 & 0 & k_0 & k_1 & k_2 & \ldots \\ 0 & 0 & 0 & 0 & k_0 & k_1 & \ldots \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

Stability of {X_n}

- Stability ⟺ $\rho < 1$; intuitively,
- Stability ⟺ average number of customers who arrive during one service time, E[A], is less than 1

$$
\begin{aligned}
E[A] &= \sum_{n=0}^{\infty} nP[A=n] = \sum_{n=0}^{\infty} nk_n \\
&= \sum_{n=0}^{\infty} n\int_0^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}dW_s(t) \\
&= \int_0^{\infty} e^{-\lambda t}\sum_{n=0}^{\infty}\frac{n(\lambda t)^n}{n!}dW_s(t) \\
&= \int_0^{\infty} e^{-\lambda t}\underline{(\lambda t)e^{\lambda t}}dW_s(t) \\
&= \int_0^{\infty}\lambda t dW_s(t) = \lambda\int_0^{\infty} t dW_s(t) = \lambda W_s \\
&= \lambda/\mu = \rho
\end{aligned}
$$

- If $\rho < 1$, the embedded Markov chain {X_n} is ergodic (proof see P*304*) and thus has a steady-state probability distribution π .

It's shown that: ρ=1, {X_n} is recurrent null; ρ>1, {X_n} is transient. In either case, {X_n} has no steady-state distribution!

Summary

- {*N(t), t* ≥ *0*} representing the number of customers in the system at time *t* is NOT a Markov process
- An embedded Markov chain can be defined by explaining *N(t)* only at instants of departure
  - $X_n = N(t_n)$: number of customers the n^{th} departing customer leaves behind
  - { X_n } is a Markov chain

$$
P = [P_{ij}] = \begin{bmatrix}
k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \cdots \\
k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \cdots \\
0 & k_0 & k_1 & k_2 & k_3 & k_4 & \cdots \\
0 & 0 & k_0 & k_1 & k_2 & k_3 & \cdots \\
0 & 0 & 0 & k_0 & k_1 & k_2 & \cdots \\
0 & 0 & 0 & 0 & k_0 & k_1 & \cdots \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & .
\end{bmatrix}
$$

  - {X_n} is ergodic and thus has a steady-state probability distribution π when ρ= E[A] < 1

How to find π_i?

π_i = steady state probability that a departing customer leaves *i* customers behind.

Find π

- We will be interested in two discrete distributions and their *z*-transforms
  - π_i = steady state probability that a departing customer leaves *i* customers behind = Pr{X=i}
  - k_i = probability that i customers arrive during a service time interval = Pr{A=i}

z-transforms (*review*):

$$Defn:\ g_Y(z) = E[z^Y] = \sum_{i=0}^{\infty} y_i z^i$$

$$\Downarrow$$

$$g_X(z) = \pi(z) = \sum_{i=0}^{\infty} \pi_i z^i$$

$$g_A(z) = k(z) = \sum_{i=0}^{\infty} k_i z^i$$

Find π (Cont'd)

- $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ is the solution to the equations

$$\begin{cases} \pi = \pi P \\ \sum_{i=0}^{\infty} \pi_i = 1 \end{cases}$$

$$P = [P_{ij}] = \begin{bmatrix} k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \ldots \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & \ldots \\ 0 & k_0 & k_1 & k_2 & k_3 & k_4 & \ldots \\ 0 & 0 & k_0 & k_1 & k_2 & k_3 & \ldots \\ 0 & 0 & 0 & k_0 & k_1 & k_2 & \ldots \\ 0 & 0 & 0 & 0 & k_0 & k_1 & \ldots \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

$$P_{ij} = \begin{cases} k_{j-i+1}, & j \ge i-1, i \ge 1 \\ 0, & j < i-1, i \ge 1 \end{cases}$$

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Find π (Cont'd)

- Examine the component by component statement of matrix equation $\pi = \pi P$:

$$\boxed{\pi_i = \sum_{j=0}^{\infty} \pi_j P_{ji}, i = 0,1,2,\ldots}$$

$$\pi_i = \sum_{j=0}^{\infty} \pi_j P_{ji} = \pi_0 P_{0i} + \sum_{j=1}^{i+1} \pi_j P_{ji} + 0$$

$$= \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1}$$

Previous customer left 0 behind

i customers arrived during service

Previous customer left *j* behind

i-j+1 arrived during service

Multiplying by z^i:

$$\pi_i z^i = \pi_0 k_i z^i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1} z^i$$

$$= \pi_0 k_i z^i + \frac{1}{z} \sum_{j=1}^{i+1} \pi_j k_{i-j+1} z^{i+1}$$

$$= \pi_0 k_i z^i + \frac{1}{z} \sum_{j=0}^{i+1} \pi_j k_{i-j+1} z^{i+1} - \frac{\pi_0 k_{i+1} z^{i+1}}{z}$$

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Find π (Cont'd)

Summing over *i*:

$$\sum_{i=0}^{\infty} \pi_i z^i = \pi(z)$$

$$= \sum_{i=0}^{\infty}\left[\pi_0 k_i z^i + \frac{1}{z}\sum_{j=0}^{i+1} \pi_j k_{i-j+1} z^{i+1} - \frac{\pi_0 k_{i+1} z^{i+1}}{z}\right]$$

$$= \pi_0 \sum_{i=0}^{\infty} k_i z^i + \frac{1}{z}\sum_{i=0}^{\infty}\sum_{j=0}^{i+1} \pi_j k_{i-j+1} z^{i+1} - \frac{\pi_0}{z}\sum_{i=0}^{\infty} k_{i+1} z^{i+1}$$

$$= item\ \#1 + item\ \#2 + item\ \#3$$

Where,

$$item\ \#1 = \pi_0 \sum_{i=0}^{\infty} k_i z^i = \pi_0 k(z)$$

$$item\ \#3 = -\frac{\pi_0}{z}\sum_{i=0}^{\infty} k_{i+1} z^{i+1} = -\frac{\pi_0}{z}\left[\sum_{i=0}^{\infty} k_i z^i - k_0 z^0\right]$$

$$= -\frac{\pi_0}{z}\left[k(z) - k_0\right]$$

$$item\ \#2 = \frac{1}{z}\sum_{i=0}^{\infty}\sum_{j=0}^{i+1} \pi_j k_{i-j+1} z^{i+1}$$

$$= \frac{1}{z}\left[k(z)\pi(z) - k_0 \pi_0\right]$$

Proof

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Find π (Cont'd)

- Summing all terms

  $\pi(z)$ = term #1 + term #2 + term #3

  $= \pi_0 k(z) + 1/z\, \pi(z)\, k(z) -$

  $1/z\, \pi_0 k_0 - 1/z\, \pi_0 k(z) + 1/z\, \pi_0 k_0$

  By rearranging:

  $\pi(z)\, [\,1 - 1/z\, k(z)] = \pi_0 [\,1 - 1/z]\, k(z)$

  Multiplying both sides by (-z)

  $\pi(z)\, [k(z) - z] = \pi_0 [1 - z]\, k(z)$

$$\boxed{\pi(z) = \frac{\pi_0 (1-z) k(z)}{k(z) - z}} \qquad \text{(Eq\#1)}$$

Note:

$$\pi(1) = \sum_{i=0}^{\infty} \pi_i = 1$$

$$k(1) = \sum_{i=0}^{\infty} k_i = 1$$

$$E[A] = \sum_{n=0}^{\infty} n k_n = k'(1) = \rho$$

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Find π (Cont'd)

$$1=\pi(1)=\lim_{z\to 1}\pi(z)=\lim_{z\to 1}\frac{\pi_0(1-z)k(z)}{k(z)-z}$$

Applying L'Hopital's rule:

$$=\lim_{z\to 1}\frac{[\pi_0(1-z)k(z)]'}{[k(z)-z]'}$$

$$=\lim_{z\to 1}\frac{\pi_0[(1-z)k'(z)-k(z)]}{k'(z)-1}$$

$$=\frac{-\pi_0 k(1)}{k'(1)-1}=\frac{-\pi_0}{\rho-1}=\frac{\pi_0}{1-\rho}$$

$$\Rightarrow \quad \pi_0=1-\rho$$

- The steady-state probability distribution is given by:

$$\begin{cases}\pi_0=1-\rho\\ \pi_i=\pi_0 k_i+\sum_{j=1}^{i+1}\pi_j k_{i-j+1}\end{cases}$$

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Interpretation of Results

- π_i : steady state probability that a departing customer leaves i customers behind – π_i = *Pr[X=i]*
- p_i : probability that there are i customers in the system at arbitrary times – p_i = *Pr[N=i]*
- r_i : probability that an arriving customer finds i customers already in the system

- It can be shown (by Klienrock) that for M/G/1 systems:

$$\pi_i=p_i=r_i$$

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Interpretation of Results (Cont'd)

- Using $\pi_0 = 1 - \rho$, Eq#1 becomes:

$$\boxed{\begin{aligned}\pi(z) &= \frac{(1-\rho)(1-z)k(z)}{k(z)-z} \\ &= p(z) = r(z)\end{aligned}} \qquad \text{(Eq\#2)}$$

which is the z-transform of the steady-state probability distribution!

Agenda (M/G/1)

- Embedded Markov-chain $\{X_n\}$ solution to analyzing M/G/1 systems
  - ✓ Transition probability matrix P
  - ✓ System stability: E[A]=ρ<1
  - ✓ Steady-state probability distribution:

$$\begin{cases} \pi_0 = 1 - \rho & \text{(same as for M/M/1)} \\ \pi_i = \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1} \end{cases}$$

$$\pi_i = p_i = r_i$$

  - **Find performance measures**
    - L: average number of customers in the system
    - W: average response time
    - L_q: average number of customers in the queue (= average queue length)
    - W_q: average waiting time

Find L

- According to the "moment generating property" of the z-transform:
  - Average number of customers in the system: L = E[N] = p`(1) = π`(1)

$$\pi(z) = \frac{(1-\rho)(1-z)k(z)}{k(z)-z} = p(z) = r(z)$$

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Find L (Cont'd)

- Differentiating Eq #2

$$p`(z) = \pi'(z) = \left[\frac{(1-\rho)(1-z)k(z)}{k(z)-z}\right]' = \frac{\overbrace{(1-\rho)\left[-k^2(z) - zk'(z) + z^2k'(z) + k(z)\right]}^{u}}{\underbrace{(k(z)-z)^2}_{v}}$$

- Evaluating for z = 1 using L'Hopital's rule:

$$L = \pi'(1) = \lim_{z\to 1}\pi'(z) = \lim_{z\to 1}\frac{u}{v} = \lim_{z\to 1}\frac{u'}{v'} \Rightarrow$$

$$\mathrm{L} = \rho + \frac{\mathrm{k}''(1)}{2(1-\rho)} \qquad \text{(Eq \#3)}$$

Need to find k``(1)!

Using LST it can be shown (extra notes):

$$k(z) = W_s^*[\lambda - \lambda z], \quad W_s^*[.] \text{ is LST of } s$$
$$k'(z) = -\lambda W_s^{*(1)}[\lambda - \lambda z],$$
$$k''(z) = \lambda^2 W_s^{*(2)}[\lambda - \lambda z] \qquad \text{(Eq \#4)}$$

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Find L (cont'd)

- Hence based on (Eq #4):

$$k'(1) = -\lambda W_s^{*(1)}[0]$$
$$k''(1) = \lambda^2 W_s^{*(2)}[0]$$

- By MGP of LST → $E[X^n] = (-1)^n \left.\frac{d^n X^*[\theta]}{d\theta^n}\right|_{\theta=0}$:

$$W_s = E[s] = -\left.\frac{dW_s^*[\theta]}{d\theta}\right|_{\theta=0} = -W_s^{*(1)}[0]$$
$$E[s^2] = (-1)^2 \left.\frac{dW_s^{*(2)}[\theta]}{d\theta}\right|_{\theta=0} = W_s^{*(2)}[0]$$

- We have:

$$k'(1) = -\lambda W_s^{*(1)}[0] = -\lambda(-W_s) = \rho$$
$$k''(1) = \lambda^2 W_s^{*(2)}[0] = \lambda^2 E[s^2]$$

- Substituting into (Eq #3)

$$\mathrm{L} = \rho + \frac{\mathrm{k}''(1)}{2(1-\rho)} = \rho + \frac{\lambda^2 E[s^2]}{2(1-\rho)}$$

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Find L (cont'd)

$$\mathrm{L} = \rho + \frac{\lambda^2 E[s^2]}{2(1-\rho)}, \text{ using } \mathrm{E}[s^2] = \mathrm{Var}[s] + \mathrm{E}^2[s]:$$

$$\mathrm{L} = \rho + \frac{\lambda^2(\mathrm{Var}[s] + \mathrm{E}^2[s])}{2(1-\rho)} = \rho + \frac{\lambda^2 \mathrm{Var}[s] + \rho^2}{2(1-\rho)}$$

$$= \rho + \frac{\rho^2(1 + C_s^2)}{2(1-\rho)}, \text{ where } C_s = \frac{\sqrt{Var[s]}}{E[s]} \text{ is C.O.V.}$$

C.O.V: Coefficient of Variation

-- the Pollaczek-Khintchine formula

The P-K formula shows how the expected number of customers in the M/G/1 system depends on C_s

Note: for the exponential service time distribution: C_s=1, then

$$\mathrm{L} = \rho + \frac{\rho^2(1+1)}{2(1-\rho)} = \rho + \frac{\rho^2}{1-\rho} = \frac{\rho}{1-\rho}$$

which is the same as in M/M/1!

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Agenda (M/G/1)

- Embedded Markov-chain $\{X_n\}$ solution to analyzing M/G/1 systems
  - ✓ Transition probability matrix P
  - ✓ System stability: E[A]=ρ<1
  - ✓ Steady-state probability distribution:

$$\begin{cases} \pi_0 = 1-\rho & \text{(same as for M/M/1)} \\ \pi_i = \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1} \end{cases}$$

$$\pi_i = p_i = r_i$$

  – Find performance measures
    - ✓ L: average number of customers in the system
    - W: average response time
    - L_q: average number of customers in the queue (= average queue length)
    - W_q: average waiting time

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Find W, L_q, W_q

- W (average response time):

  By Little's Law:

$$W = L/\lambda = \frac{\rho + \frac{\rho^2(1+C_s^2)}{2(1-\rho)}}{\lambda}$$

- L_q (average queue length):

$$\begin{aligned} L_q &= L-(1 * P[\text{Server is not empty}] \\ &= L-(1 - P[0 \ \text{ customer in the system}]) \\ &= L-(1-\pi_0) = L-(1-(1-\rho)) \\ &= L-\rho = \frac{\rho^2(1+C_s^2)}{2(1-\rho)} \end{aligned}$$

- W_q (average waiting time):

$$W_q = L_q/\lambda = \frac{\rho^2(1+C_s^2)}{2(1-\rho)\lambda} = \frac{\rho W_s(1+C_s^2)}{2(1-\rho)}$$

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Another Way to Find L_q, W_q

- Notations:
  - *s*: r.v. describing the service time
  - *q*: r.v. describing the time a customer spends in the queue before service begins
  - *w*: r.v. describing the total time a customer spends in the system: $w = q+s$
- **Takacs Recurrence Theorem**
  - To calculate moment of queueing time $E[q^i]$, in terms of moment of service time $E[s^i]$
  - Given an M/G/1 system in which $E[s^{j+1}]$ exists, then $E[q]$, $E[q^2]$, ……, $E[q^j]$ also exist and

$$E[q^k] = \frac{\lambda}{1-\rho}\sum_{i=1}^{k}\binom{k}{i}\frac{E[s^{i+1}]}{i+1}E[q^{k-i}] \qquad k = 1,2,\ldots, j$$

where $E[q^0]=1$

  - *Corollary*: then the moments $E[w]$, $E[w^2]$, ……, $E[w^j]$ also exist and

$$E[w^k] = \sum_{i=0}^{k}\binom{k}{i}E[s^i]E[q^{k-i}], \quad k = 1,2,\ldots, j$$

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Another Way to Find L_q, W_q(Cont'd)

By "Takacs Recurrence Theorem":

$$\begin{aligned}
W_q = E[q] &= \frac{\lambda}{1-\rho}\sum_{i=1}^{1}\binom{1}{i}\frac{E[s^{i+1}]}{i+1}E[q^{1-i}] \\
&= \frac{\lambda}{1-\rho}\frac{E[s^2]}{2}E[q^0] \\
&= \frac{\lambda}{1-\rho}\frac{Var[s]+E^2[s]}{2} \\
&= \frac{\lambda}{1-\rho}\frac{E^2[s]C_s^2+E^2[s]}{2} \\
&= \frac{\lambda E^2[s](1+C_s^2)}{2(1-\rho)} \\
&= \frac{\lambda W_s W_s(1+C_s^2)}{2(1-\rho)} = \frac{\rho W_s(1+C_s^2)}{2(1-\rho)}
\end{aligned}$$

which agrees with the previous result

By Little's Law:

$$L_q = W_q\lambda = \frac{\rho W_s(1+C_s^2)}{2(1-\rho)}\lambda = \frac{\rho^2(1+C_s^2)}{2(1-\rho)}$$

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Another Way to Find L, W (Cont'd)

By the Corollary:

$$W = E[w] = \sum_{i=0}^{1} \binom{1}{i} E[s^i]E[q^{1-i}]$$
$$= E[s^0]E[q^1] + E[s^1]E[q^0]$$
$$= E[q] + E[s]$$
$$= \frac{\rho W_s(1+C_s^2)}{2(1-\rho)} + W_s$$
$$= \frac{2\rho(1-\rho) + \rho^2(1+C_s^2)}{2\lambda(1-\rho)}$$

which agrees with the previous result

By Little's Law:

$L = W\lambda$

M/D/1 Queueing Systems

- Assume
  - Poisson arrival process with rate λ
  - Deterministic service rate μ
    - Constant service time s = W_s = 1/ μ
- A special case of M/G/1 with Var[s] = 0, E[s] = W_s → C.O.V.

$$C_s = \frac{\sqrt{Var[s]}}{E[s]} = 0$$

M/D/1 Queueing Systems (cont'd)

- Performance measures

$$L = \rho + \frac{\rho^2(1+C_s^2)}{2(1-\rho)} = \rho + \frac{\rho^2}{2(1-\rho)} = \frac{\rho(2-\rho)}{2(1-\rho)}$$

$$W = L/\lambda = \frac{\rho(2-\rho)}{2(1-\rho)\lambda} = \frac{W_s(2-\rho)}{2(1-\rho)}$$

$$L_q = \frac{\rho^2(1+C_s^2)}{2(1-\rho)} = \frac{\rho^2}{2(1-\rho)}$$

$$W_q = L_q/\lambda = \frac{\rho^2}{2(1-\rho)\lambda} = \frac{\rho W_s}{2(1-\rho)}$$

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Topics

- Transform Methods
- M/G/1, M/D/1
- **GI/M/1**

Solution:

- Constructing an embedded Markov chain
- And applying z-transform and Laplace-Stieltjes transform methods

"Embedded Markov Chain Queueing Systems"

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Renewal Processes (Chapter 4.5)

- A Poisson process can be characterized as a counting process for which the inter-arrival times (times between successive events) are *i.i.d*, exponential *r.v.*s
- A renewal process is a generalization of the Poisson process

Renewal Processes

- Let $\{N(t), t >= 0\}$ be a counting process
- X_1: the time of occurrence of the first event
- X_n: the time between the (n-1)*th* and the n*th* event of the process for $n >= 2$
- If the sequence of nonnegative *r.v.*s $\{X_n, n >= 1\}$ is *i.i.d.*, then $\{N(t), t >= 0\}$ is a renewal (counting) process
  - Common c.d.f.

$$F(x) = P[X_n \le x], n = 1,2,3,......$$

    - F(0)=0
  - Common mean: μ
  - Common variance: σ^2

Renewal Processes (Cont'd)

- A renewal has taken place when an event counted by N(t) occurs
  - N(t): number of renewals in the interval (0,t]
  - M(t)=E[N(t)]
    - mean number of renewals in (0,t]
    - Called "renewal function"
- The waiting time until the *n*th renewal (W_n):

$$W_0 = 0, \quad W_n = X_1 + X_2 + ... + X_n, n \geq 1$$

  - $\{W_n, n \geq 0\}$ is also called the renewal process.

Renewal Processes (Cont'd)

- Elementary Renewal Theorem:
  - Let {N(t), t >= 0} be a renewal process with $E[X_n] = \mu$ for all *n*. Then:

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}$$

  - Proposition:

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$$

Renewal Processes (Cont'd)

- Example 1: A light bulb is installed at time $W_0=0$. When it burns out at time $W_1=X_1$, it is replaced by a new bulb, which burns out at time $W_2=X_1+X_2$.
- This process continues indefinitely: as each bulb burns out, it is replaced with a brand new one.
- Assume the successive bulb lifetimes

$$\{X_n, n \geq 1\}$$

are *i.i.d.*; N(t) is the number of bulb replacements that occur by time t. Then

$$\{N(t), t \geq 0\}$$

is a renewal process.

Renewal Processes (Cont'd)

- Example 2: Suppose the renewal process $\{N(t), t \geq 0\}$ is Poisson with parameter λ. Then,

$$P[N(t)=k]=e^{-\lambda t}\frac{(\lambda t)^k}{k!}, k=0,1,2...$$

$$M(t)=E[N(t)]=\lambda t$$

Poisson process is the only renewal process with a linear renewal function!

- Suppose $\{N(t), t \geq 0\}$ is a renewal process with renewal function *M(t)=5t*. What is the probability distribution of the number of renewals by time t=*15*? What is the probability that there are 100 renewals by time 15?

GI/M/1 Queueing Systems

- Assume
  - Renewal arrival process
    - The inter-arrival times are i.i.d. *r.v.*s
  - Exponential service time with a mean of $1/\mu$
    - μ is the average service rate

GI/M/1 Queueing Systems

- $\{N(t), t \geq 0\}$ representing the number of customers in the system at time t is NOT a Markov process

- However, by explaining N(t) only at instants of arrival, we can define an embedded Markov chain
  - Let $0<t_1<t_2<\ldots<t_n<\ldots$ denote the successive times at which a customer arrives
  - $X_n = N(t_n)$: number of customers the n^{th} arriving customer finds in the system
  - $\{ X_n \}$ is a Markov chain

GI/M/1 (Cont'd)

- If $\rho < 1$, the embedded Markov chain $\{X_n\}$ is ergodic and thus has a steady-state probability distribution $\pi = \{\pi_n\}$.
- π_n: the steady-state probability that an arriving customer finds *n* customers in the system, for n = 0, 1, 2, ...
- X: *r.v.* describing the number of customers that an arriving customer finds in the system, thus, $\pi_n = P[X=n]$

GI/M/1 (Cont'd)

- Wolff showed that
  - π_0 is the unique solution of equation

$$\boxed{1-\pi_0 = A^*[\mu\pi_0]}$$

such that $0 < \pi_n < 1$, where $A^*[\theta]$ is the LST of the inter-arrival time τ

  - General expression of π_n in terms of π_0:

$$\boxed{\pi_n = \pi_0(1-\pi_0)^n, n = 0,1,2,...}$$

A geometric distribution (L#6) with $q=1-\pi_0$, $p=\pi_0$, thus,

$$E[X] = \frac{q}{p} = \frac{1-\pi_0}{\pi_0}, Var[X] = \frac{q}{p^2} = \frac{1-\pi_0}{\pi_0^2}$$

Applications (1)

Consider the **M/M/1** queueing system

- First, find **A*[θ]:**
  - the inter-arrival time τ is exponentially distributed with rate λ, that is,

$$\Pr[\tau \le t] = 1 - e^{-\lambda t}$$

$$pdf : f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Second, find π_0

$$\boxed{1 - \pi_0 = A^*[\mu\pi_0]}$$

- Third, find π_n

$$\boxed{\pi_n = \pi_0(1 - \pi_0)^n}$$

Applications (2)

Consider the **D/M/1** queueing system

- First, find **A*[θ]:**
  - the inter-arrival time τ is a constant 1/λ

- Second, find π_0

$$\boxed{1 - \pi_0 = A^*[\mu\pi_0]}$$

- Third, find π_n

$$\boxed{\pi_n = \pi_0(1 - \pi_0)^n}$$

GI/M/1 (Cont'd)

- Distinction between
  - π_n: the steady-state probability that an arriving customer finds *n* customers in the system
    - From "an arriving customer" point of view
  - p_n: the steady-state probability that there are *n* customers in the system
    - From "a random observer" point of view

An Illustrating Example (π_n *vs* P_n)

- A D/D/1 with E[τ]=10 min, W_s=5 min → ρ=1/2
  - E[τ] > W_s → an arriving customer never sees another customer, hence

$$\pi_0 = 1, \pi_n = 0 \text{ for } n \geq 1$$

  - ρ=1/2 → the server is busy half of the time, i.e., the system contains one customer half the time and is empty half the time as observed by an outside observer, hence

$$p_0 = 0.5, p_1 = 0.5, p_n = 0 \text{ for } n \geq 2$$

$\pi_n = p_n$ *iff* the arrival process is Poisson (shown by Wolff)

GI/M/1 (Cont'd)

- **p_n ?**

- Kleinrock showed that for GI/M/1:

$$p_0 = 1 - \rho$$
$$p_n = \rho\pi_0(1-\pi_0)^{n-1}, n = 1,2,...$$

GI/M/1 (Cont'd)

- Find L, W, L_q, W_q?

$$L = \sum_{n=0}^{\infty} np_n = \rho\pi_0 \sum_{n=0}^{\infty} n(1-\pi_0)^{n-1}$$
$$= \rho\pi_0 \left(\sum_{n=0}^{\infty}(1-\pi_0)^n\right)' = \rho\pi_0 \frac{1}{\pi_0^2} = \frac{\rho}{\pi_0}$$

$$L_q = L - (1 * P[\text{Server is not empty}]$$
$$= L - (1 - P[0 \text{customer in the system}])$$
$$= L - (1 - p_0) = L - (1 - (1 - \rho))$$
$$= L - \rho = \frac{\rho}{\pi_0} - \rho = \frac{\rho(1-\pi_0)}{\pi_0}$$

$$W = L / \lambda = \frac{\rho}{\pi_0} / \lambda = \frac{W_s}{\pi_0}$$
$$W_q = L_q / \lambda = \frac{(1-\pi_0)W_s}{\pi_0}$$