ECE560: Computer Systems Performance Evaluation

Lecture #10 –
Markov Processes

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Administration Issues (2/21, Wed.)

- Homework #3
  - Due: **Today**
- Project proposal (refer to Guidelines)
  - Due: **February 23, Friday**
- Today's topics
  - Finish L#9 (birth-death process)
  - Then L#10 (Markov process)

Review of Lecture #9

- Stochastic processes: a family of r.v.'s {X(t) | t∈ T} that is indexed by a parameter t
  1. Discrete **time**, stochastic **chain**
  2. Continuous time, stochastic **chain**
  3. Discrete **time**, continuous state process
  4. Continuous **time**, continuous state process
- Important stochastic processes
  – Counting process: a SP representing the number of events of some kind that have occurred by the time t; Continuous parameter, discrete state;
  – Poisson process: a special counting process satisfying *independent Increments & stationary increments* properties
  – Birth-depth process: steady-state solution (balance equations)

Topics

- **Markov processes**

Chapter 4 in Allen's book

Markov Processes

A stochastic process is a Markov process

iff

probabilities of future states depend only on the current state and not on how it reached that state

i.e., all influences of the past on the system's future is contained in the current state.

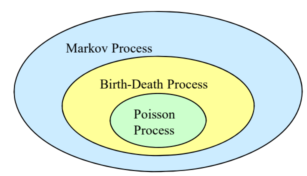
Formal Definition

A stochastic process $\{X(t), t \in T\}$ is a Markov process if for any set of $n + 1$ values $t_1 < t_2 < \ldots < t_n < t_{n+1}$ in the index set T and any set of n+1 states $\{ x_1, x_2, \ldots, x_n , x_{n+1}\}$

$$P[\, X(t_{n+1})=x_{n+1} | X(t_1)=x_1, X(t_2)=x_2, \ldots X(t_n)=x_n]$$

$$\equiv P[\, X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n]$$

Note: All birth-and-death processes are Markov Processes. Hence, the Poisson process is a Markov Process



Definitions

- Markov chains
- Discrete-time Markov chains
- Homogeneous Markov chains

Markov Chains

- A discrete state Markov process is called a Markov chain.
- States can be labeled { E_0, E_1, E_2,. . . }
- Typically the states are just denoted using non-negative integers { 0, 1, 2, . . . }, where state i corresponds with E_i.

- A discrete time Markov chain makes state transitions at times t_n, n = 1, 2, 3, . . . (possibly into the same state)

Notation: $\{ X(t_n), \; n = 0, 1, 2, \ldots \} \rightarrow \{ X_n \}$

One-step state transition probabilities:

$$P[X_{n+1} = j \mid X_n = i], \; n, i, j = 0, 1, 2, \ldots$$

Homogeneous Markov Chains

Markov chains for which the one-step transition probabilities are independent of time index *n*, i.e. ,

$$P[X_{n+1} = j \mid X_n = i] = P[X_{m+1} = j \mid X_m = i]$$

$$= P_{ij} \qquad \forall\ m, n, i, j$$

Such a MC is said to have stationary transition probabilities or to be homogeneous in time.

State Transition Probability Matrix (One-Step)

$$P = [P_{ij}] = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \ldots P_{0j} \ldots \\ P_{10} & P_{11} & P_{12} & \ldots P_{1j} \ldots \\ : & : & : & : \\ P_{i0} & P_{i1} & P_{i2} & \ldots P_{ij} \ldots \\ : & : & : & : \end{bmatrix}$$

- P is a $n \times n$ matrix if the number of states is finite (n)

Otherwise,

P is an infinite matrix.

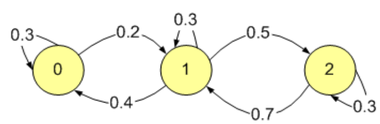
- Requirements:

$P_{ij} \geq 0 \qquad i, j = 0, 1, 2, \ldots$

$$\sum_{j=0}^{n/\infty} P_{ij} = 1, \quad i = 0, 1, 2, \ldots\ldots$$

Hands-On Problem

- Consider a discrete-time "Markov chain" expressed by the following state transition diagram



a) Set up the one-step state transition probability matrix P of the "Markov chain".
b) Is this a valid Markov chain? If not, make the least changes without adding more transitions to make the Markov chain become valid.

Discrete-Time MC: Example I

- Consider a sequence of Bernoulli trials, for each trial
  - Success probability is p
  - Failure probability is $q = 1 - p$

  Assume X_n -- the state of the process at trial n, is the number of uninterrupted successes that have been completed at this point, i.e., the length of consecutive successes.

  Find the state transition probability matrix and state transition diagram.

Trial	**:**	**F**	**S**	**S**	**F**	**F**	**S**	**S**	**S**	**F**
n =		**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\mathbf{X_n}$ =										

State Transition Probability Matrix (n-Step)

- n-step transition probabilities

$$P_{ij}^{(n)} = P[X_n = j \mid X_0 = i]$$

For homogeneous MC

$$P_{ij}^{(n)} = P[X_{m+n} = j \mid X_m = i], \quad \forall m \geq 0, n > 0$$

And

$$P_{ij}^{(1)} = P_{ij}$$

$$P_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{Kronecker Delta function}$$

- Denote the matrix of n-step transition probabilities by

$$P^{(n)} = [P_{ij}^{(n)}]$$

Then

$$P^{(n)} = P^{(n-1)}P = P^n$$

State Transition Probability Matrix (n-Step) (Proof)

- Why $P^{(n)} = P^{(n-1)}P = P^n$?

$$P_{ij}^{(n+m)} = P\{X_{n+m} = j | X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k | X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j | X_n = k, X_0 = i\} P\{X_n = k | X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)} \qquad \forall n, m, i, j \geq 0$$

-- Chapman-Kolmogorov equations

In particular, when $m=1$

$$P_{ij}^{(n+1)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj} \qquad n = 1,2,\ldots, \ \forall i, j \geq 0$$

$$i.e., P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik}^{(n-1)} P_{kj} \quad n = 2,3,\ldots, \ \forall i, j \geq 0$$

$$P^{(n)} = [P_{ij}^{(n)}] \Rightarrow P^{(n)} = P^{(n-1)}P = P^n$$

Markov Chains (Cont'd)

- Let $\Pi_j(n)$ represent the probability of being in state j at the *n*th step (transition), i.e.

$$\Pi_j(n) = P[X_n = j]$$

- In vector notation:

$$\Pi(n) = (\Pi_0(n), \Pi_1(n), \ldots \Pi_{m-1}(n))$$

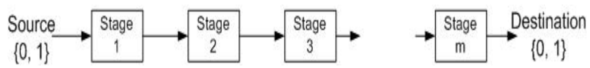
One-step transition probability matrix:

$$P_{m\times m} = [P_{ij}] = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \ldots & P_{0m-1} \\ P_{10} & P_{11} & P_{12} & \ldots & P_{1m-1} \\ : & : & : & & : \\ P_{m-10} & P_{m-11} & P_{m-12} & \ldots & P_{m-1m-1} \end{bmatrix}$$

$$\Pi(n) = \Pi(n-1) \times P$$
$$\Pi(n) = \Pi(0)\, P^n$$

Discrete-Time MC: Example II

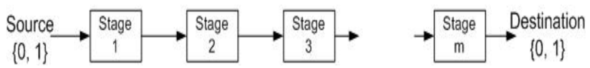
A communication system transmits the digit 0 and 1 through several stages. At each stage, there is a probability of 0.75 that the output will be the same digit as the input.



What is the probability that a 0 that is entered at the first stage is output as a 0 from the 4th stage?

Review Question

A communication system transmits the digit 0 and 1 through several stages. At each stage, there is a probability of 0.75 that the output will be the same digit as the input.



What is the probability that a 0 that is entered at the first stage is output as a 1 from the 3rd stage?

Irreducible Markov Chains

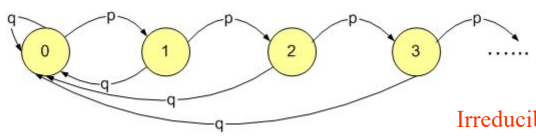
- State j of a Markov chain $\{X_n\}$ is said to be reachable from state i if it is possible for the chain to proceed from state i to state j in a finite number of transitions, i.e.

$$P_{ij}^{(n)} > 0, \quad \text{for some } n \geq 0$$

- Two states i and j are to communicate if i is reachable from j and j is reachable from i.
- If every state is reachable from every other state, the chain is said to be irreducible.

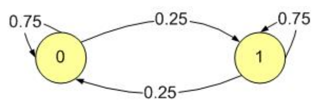
Examples

- Bernoulli trials example



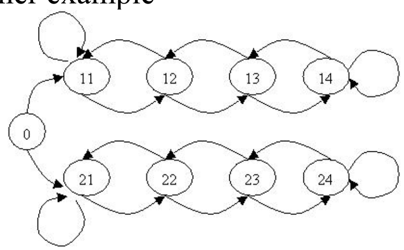
Irreducible!

- Communication system example



Irreducible!

- Another example



Reducible!

Irreducible Markov Chain?

- Can tell easily from the state transition diagram
- Given the probability transition matrix P, take n-th power of P, if for some n, a positive matrix (all items >0) is obtained, which means that a transition can be made between any two states in n steps → Markov chain is irreducible.
  - Example *4.4.4* in Allen

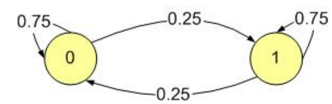
Aperiodic Markov Chains

- The period of state i, $d(i)$, is the greatest common divisor of the set of all positive integers n such that $P_{ii}^{(n)} > 0$
  - If $P_{ii}^{(n)} = 0, \forall n \geq 1$, define d(i) = 0
  - If $d(i) > 1$, state i is said to be periodic
  - If $d(i) = 1$, state i is said to be aperiodic
    - A state i for which $P_{ii} > 0$ has period of 1

- A Markov chain is aperiodic if every state has period 1 (is aperiodic)

Examples

- Every state is aperiodic



- Every state is periodic with period 2

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Recurrent Markov Chains

- For each state i, define $f_i^{(n)}$ to be the probability that the first return to state i occurs, n steps (transitions) after leaving i, i.e.,

$$f_i^{(n)} = P[X_n = i, X_k \neq i \text{ for } k = 1,2,\ldots,n-1 \mid X_0 = i]$$

Define: $f_i^{(0)} = 1, \forall i$

Then, the probability of ever returning to state i is given by

$$f_i = \sum_{n=1}^{\infty} f_i^{(n)}$$

- If $f_i < 1$, then state i is a transient state
- If $f_i = 1$, then state i is a recurrent state
  - Mean recurrence time of i, i.e., the average time (steps) to return to state i is

$$m_i = \sum_{n=1}^{\infty} n f_i^{(n)}$$

  - If $m_i = \infty$, state i is said to be *recurrent null*
  - If $m_i < \infty$, state i is said to be *positive recurrent* or recurrent non-null.

Ergodic Markov Chains

- A discrete-time Markov chain is said to be **ergodic** if it is
  - irreducible: you can get from every state to every other
  - aperiodic: every state has period 1. For each state there are paths back to that state of various lengths, i.e., not all multiples of the same integer k>1.
  - for which all states are positive recurrent: for each state, upon leaving the state you will return with probability 1 and within a finite mean time.

A finite-state Markov chain that is irreducible and aperiodic is ergodic.

Probability Distribution of MC

- Let $\Pi_j(n)$ represent the probability that discrete time Markov chain $\{X_n\}$ is in state j at the nth step (transition), i.e. $\Pi_j(n) = P[X_n = j]$
- Initial distribution of state j is $\Pi_j(0) = P[X_0 = j]$, j=0,1,…

- A discrete Markov chain is said to have a *stationary probability distribution* $\Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots)$ if
  - $\Pi_j(0) = \Pi_j(n) = \Pi_j$, $\forall_j \ \forall_n$
  - Equivalently, the matrix equation $\Pi = \Pi^*P$ is satisfied.
  - Requirements

    $\Pi_i \geq 0 \ \forall_i$ and $\sum_i \Pi_i = 1$

- A Markov chain is said to have a *long-run* or *limiting probability distribution* $\Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots)$ if

$$\lim_{n\to\infty} \Pi_j(n) = \lim_{n\to\infty} P[X_n = j] = \Pi_j, \ j = 0,1,\ldots\ldots$$

Properties of Ergodic MC

- The limiting probabilities

$$\lim_{n\to\infty} \Pi_j(n) = \Pi_j, \ j = 0,1,\ldots\ldots$$

  always exist and are independent of the initial state distribution $\Pi(0) = (\Pi_0(0), \Pi_1(0), \Pi_2(0), \ldots)$.

- $\Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots)$ forms a stationary probability distribution and $\boldsymbol{\Pi_j = 1 / m_j}$.

- The limiting distribution is the unique solution to the equations

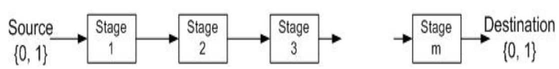
  $\Pi = \Pi^*P$ and $\sum_i \Pi_i = 1$

- Stationary probability distribution = Long-run (limiting) probability distribution

  Also called "equilibrium" or "steady-state" distribution.

Discrete-Time MC: Example II (Revisit)

A communication system transmits the digit 0 and 1 through several stages. At each stage, there is a probability of 0.75 that the output will be the same digit as the input.



What is the limiting probability that a 0 entered into the first stage is output as a 0 from the n^{th} stage as $n \rightarrow \infty$?

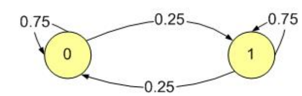
Balance Equations of MC (1)

- Balance equations

  Transition entering = Transition leaving

  $\Pi = \Pi P \Leftrightarrow$ Balance Equations

- Example
  - 2-state communication system



Transition leaving = Transition entering

Π_0 *(0.25+0.75) = Π_0 * 0.75+ Π_1 * 0.25

i.e. Π_0 * 0.25 = Π_1 * 0.25

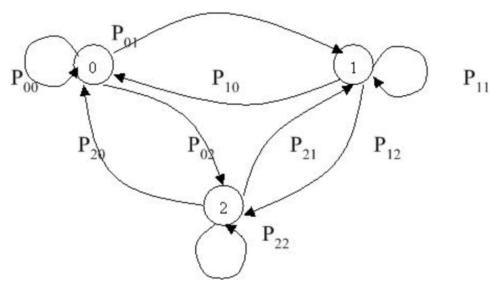
Or $\Pi_0 = \Pi_1$

This is the same as the one obtained from

$\Pi = \Pi$*P

Exercise

Write down the balance equations for the following MC:



Hands-On Problem

- Suppose that in New Zealand, home of the Gala apple, years for these wonderful apples can be described as *Great, Average,* or *Poor.* Suppose that following a *Great* year the probabilities of *Great*, *Average*, or *Poor* years are 0.5, 0.3, and 0.2, respectively. Suppose also that following an *Average* year the probabilities of *Great*, *Average*, or *Poor* years are 0.2, 0.5, and 0.3, respectively. Finally, suppose that following a *Poor* year the probabilities for *Great*, *Average*, or *Poor* years are 0.2, 0.2, and 0.6, respectively. Assume we can describe the situation from year to year by a Markov chain with the states 0, 1, and 2 corresponding to *Great*, *Average*, and *Poor* years, respectively.
  - Set up the transition probability matrix P of the Markov chain & draw the state transition diagram.
  - Is the Markov chain ergodic? Why or why not?
  - Suppose the initial probability for a *Great* year is 0.2, for an *Average* year is 0.5, and for a *Poor* year is 0.3. What is the probability of a *Great* year after **one** year?
  - What is the probability for a *Great* year after ***n*** year as $n \rightarrow$ infinity?

Next Topics

- Queueing systems

Things to Do

- Read Allen's Ch. 4