ECE560: Computer Systems Performance Evaluation

Lecture #5 –

Probability Theory & Statistics

(Part I: Review)

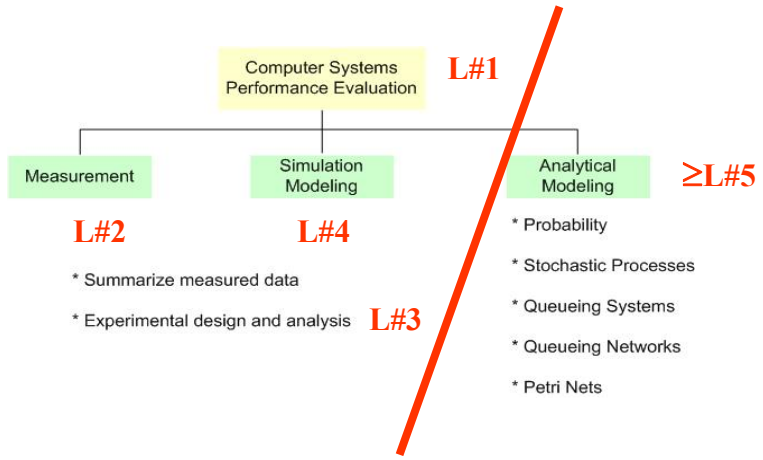
Instructor: Dr. Liudong Xing

Spring 2024

Administration Issues (2/5)

- Homework #1
  - Due: **Today**
- Homework #2
  - Please download problems from course website
  - Due: **February 12, Monday**
- Project proposal (refer to Guidelines)
  - Due: **February 23, Friday**

Lecture #1 - #4 Review



Topics (Probability Theory)

- A basic tool for modeling random / uncertain phenomena
  - Sample spaces & events
  - Axioms of probability
  - Field, σ-field, and probability measure
  - Odd
  - Conditional probability & law of total probability
  - Independence

Related reading: Allen's Ch. 2.0~2.4

Random Experiment ε

- Outcome is *unknown* in advance
- Set of all possible individual outcomes is *known*

- Example: tossing a die

Sample Space Ω

- The set of all possible individual outcomes (sample points / elementary events) of an experiment

- Example: sample space for "tossing a die"?

$$\Omega = \{1,2,3,4,5,6\}$$

- ***Exercise:*** sample space for "tossing a fair coin again and again until the first head appears"?

Sample Space Ω *(Types)*

- Finite *vs*. infinite
- Discrete *vs*. continuous
  - Discrete: sample points can be put into 1-to-1 correspondence with positive integers
    - Tossing a die
    - Tossing a fair coin again and again until the first head appears
  - Continuous: sample points consist of all numbers on some finite or infinite interval of real line
    - The computer's response time to an inquiry

Events E

- A subset of a sample space
- Example:
  - $E1$={outcome is prime}={2, 3, 5}: the event of rolling a prime number
- E may be Ω or $\varnothing$ (impossible event)
- We say <u>an event E occurs</u> if the random experiment is performed and the observed outcome ω is in E, i.e., $\omega \in \boldsymbol{E}$.

Basic operations on events (sets)

For any two events A, B, the following are also events

- $\bar{A}$ or A^c= {All outcomes not in A}
- $A \cup B$ = { All outcomes in A or B or in both}
- $A \cap B = AB$ = {All outcomes in both A and B}
  - If $AB = \varnothing$ then A and B are **mutually exclusive** (**disjoint**)
- Important properties:
  - $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$
- A – B = {all outcomes in A but not in B}
- $A \subset B$: if A occurs, so does B (A implies B)

"Axioms of Probability"

- For each event E of sample space Ω, if a number $P(E)$ is defined and satisfies the following axioms

  A1: $0 \le P(E) \le 1$

  A2: $P(\Omega) = 1$

  A3: for any sequence of **pair-wise-mutually-exclusive** events, $E_1, E_2, \ldots$ (i.e. $E_iE_j = \varnothing$ for any $i \neq j$), we have

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

-- Infinite Additivity

$P(E)$ is refer to as the **probability** of the event E

Field

- F is a collection of subsets/events of Ω
  1: F is non-empty
  2: If $A \in F \rightarrow \bar{A} \in F$
  3: If $A, B \in F \rightarrow A \cup B \in F$
  3`: If $\{A_i\}_{i=1}^{n}$, where n is finite, $A_i \in F \rightarrow \bigcup_{i=1}^{n} A_i \in F$

  F is called a **field** of subsets of Ω

- Properties
  - If $A, B \in F \rightarrow A \cap B \in F$
  - $\Omega, \varnothing \in F$
  - If $A, B \in F \rightarrow B - A = B \cap \bar{A} \in F$

σ-Field

- A σ-field (σ-algebra) F is a field with property 3` replaced by the stronger property
  3`*: If $\{A_i\}_{i=1}^{\infty}$, where $A_i \in F, \forall i = 1,2,.....$ (countably infinite) →

  $$\bigcup_{i=1}^{\infty} A_i \in F$$

  F is closed under countable union

- If F is a σ-field , then it is also closed under countable intersection

- Definition: if F is a σ-field on Ω, then (Ω, F) is called a **Measurable Space**

Probability Measure

- A **probability measure** P(.) on a σ-field F, which itself is on Ω, is nothing but a mapping/function from F → [0, 1], such that the "axioms of probability" hold.

- Other properties (Th. 2.2.1)
  - $P(\varnothing) = 0$
  - $P(\bar{A}) = 1 - P(A)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - $A \subseteq B \rightarrow P(A) \leq P(B)$

- Given a measurable space (Ω, F) and a probability measure P(.) on (Ω, F), then the triple (Ω, F, P) is called a **probability space**.

Agenda

- Probability
  - √ Sample spaces & events
  - √ Axioms of probability
  - √ Field, σ-field, and probability measure
  - – **Odd**
  - – Conditional probability & law of total probability
  - – Independence

Related reading: Allen's Ch. 2.0 ~ 2.4

Odds

- If an event A has probability P(A) of occurring, **the odds for A** are defined by the following ratio:

$$\textit{Odds} \text{ for A} = \frac{P(A)}{1 - P(A)}$$

- **The odds against A**:

$$\textit{Odds} \text{ against A} = \frac{1 - P(A)}{P(A)}$$

- If odds for A are a:b, then

$$\text{P(A)} = \frac{a}{a + b}$$

Hands-On Problem

- Suppose a bookmaker tells you the odd against "A beating B" is 3:2. Assuming the odds are correct and you are one of the bettors.
- Which one you would like to bet your money on: A or B?

Agenda

- Probability
  - √ Sample spaces & events
  - √ Axioms of probability
  - √ Field, σ-field, and probability measure
  - √ Odds
  - – **Conditional probability & law of total probability**
  - – Independence

Related reading: Allen's Ch. 2.0 ~ 2.4

Conditional Probability

- Definition: Let A and B be two events, then the **conditional probability** of A given B: P[A|B] is a number such that

1. $0 \le P[A \mid B] \le 1$,
2. $P[A \cap B] = P[B]P[A \mid B]$.

Note:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{if} \quad P(B) \neq 0$$

Hands-On Problem

- "Tossing a die": find the conditional probability of the event of the uppermost side showing 3 spots given the event of rolling an odd number occurring?

Law of Total Probability

- For any two events A, B $\in$ F,

$$P(A) = P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

- Let $\{B_i\}_{i=1}^{n}$ be a *partition* of Ω, then

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$$

Bayes' Theorem

- Based on Total Probability Theorem, we have "Bayes' Theorem / Rule / Formula"
  - Let $\{B_i\}_{i=1}^n$ be a *partition* of Ω, then for any event A with P(A)>0, we have

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{n} P(A \mid B_i)P(B_i)}$$

Hands-On Problem

- A color-blind person is chosen at random. Suppose that 6 percent of men and 0.3 percent of women are color-blind. And assume that there are an equal number of males and females.
  a. What is the probability that a randomly chosen person is color-blind?
  b. Suppose a person selected at random is found to be color-blind. What is the probability of this (color-blind) person being female?

Agenda

- Probability
  - √ Sample spaces & events
  - √ Axioms of probability
  - √ Field, σ-field, and probability measure
  - √ Odd
  - √ Conditional probability & law of total probability
  - – **Independence**

Related reading: Allen's Ch. 2.0 ~ 2.4

Independence of Two Events

- **Definition**: Events A and B are said to be independent ($A \perp\!\!\!\perp B$) if

$$P(A \cap B) = P(A) \bullet P(B)$$

- Neither event influences the occurrence of the other: P(A|B)=P(A), P(B|A)=P(B)
- Independence does NOT mean that A and B have nothing in common!
- Being independent <> being mutually exclusive

Independence of Two Events (Cont'd)

- M.E. events are independent only when at least one of them has ZERO probability!
- The independent relation is not transitive!
- If $A \perp\!\!\!\perp B$, then so are

$$\overline{A} \perp\!\!\!\perp B, \quad A \perp\!\!\!\perp \overline{B}, \quad \overline{A} \perp\!\!\!\perp \overline{B}$$

Independence of a Set of Events

A list of n events $A_1, A_2, \ldots, A_n$ is defined to be

- **pair-wise independent:** if every pair is independent
- **mutually independent:** if for each set of k ($2 \le k \le n$) distinct indices $i1,i2,\ldots,ik$, which are elements of $\{1,2,\ldots,n\}$, we have

$$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})P(A_{i2})\ldots P(A_{ik})$$

Independence of a set of events (Cont'd)

- **Mutually independent <> pair-wise independent**
- A set of events is pair-wise independent. It does not follow that the list of events is mutually independent!

Hands-on Problem

- *"tossing a fair coin three time"*
  Define the events A, B, C so that
  - A="first toss results in a tail"
  - B="second toss results in a head"
  - C=" third toss results in a tail"
- Questions: are the three events A, B, and C mutually independent? Pair-wise independent?

Next Topics

- Random Variable (*r.v.*)
  - Basic concepts
  - Discrete *r.v.s*
  - Continuous *r.v.s*
  - Jointly distributed *r.v.s*

Things to Do

- Homework
- Project proposal due **February 23, Friday**