ECE560: Computer Systems Performance Evaluation

Lecture #7

Probability Theory (Part III)

- Jointly Distributed R.V.

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Administration Issues (2/14)

- Homework #3 assigned
  - Due: **February 21, Wednesday**
- No classes on Monday, February 19 (President's Day Holiday)
  - Tuesday follows Monday's class schedule
- Project proposal (refer to Guidelines)
  - Due: **February 23, Friday**

Review of Lecture #6

- Definitions of discrete and continuous random variables
- Characteristic functions
  - Cumulative distribution function (c.d.f) for discrete and continuous *r.v.s*
  - Probability mass function (p.m.f.) for discrete *r.v.*
  - Probability density function (p.d.f.) for continuous *r.v.*
- Important parameters of *r.v.*s
  - Expectation/Mean: E[X], the *k*th moment
  - Variance/standard deviation
  - Squared coefficient of variation (C.O.V.) and C.O.V.
- Important example *r.v.*s
  - *Discrete*: Bernoulli, Binomial, Geometric, Poisson
  - *Continuous*: Uniform, Normal, Exponential

Jointly Distributed *r.v.s (Agenda)*

- Joint distribution function
- Marginal distribution function
- Joint/marginal/conditional *p.m.f.*
- Joint/marginal/conditional *p.d.f.*
- Independency
- Important parameters
- Two important properties

Related reading: Allen's Ch. 2.7

Jointly Distributed *r.v.s (1)*

- Let X and Y be two r.v.s defined on the *same* probability space (Ω, F, P). The **events [$X \le x$, $Y \le y$]** consists of all sample points $\omega \in \Omega$, such that $X(\omega) \le x$ *and* $Y(\omega) \le y$
- **Joint distribution function (J.D.F)**

$$F_{X,Y}(x, y) = P[X \le x, Y \le y], \ \forall x, y \in R$$

- **Marginal distribution function (M.D.F)**

Given $F_{X,Y}$, the individual distribution function $F_X(x)$ and $F_Y(y)$ can be computed as:

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y), \ \forall x \in R$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y), \ \forall y \in R$$

F_X and F_Y are called the M.D.F. of X and Y, respectively

Jointly Distributed *r.v.s (2)*

- Probability mass function (pmf) for discrete *r.v.s*
  - Joint pmf:

$$p(x, y) = P[X = x, Y = y]$$

  - Marginal pmf:

$$p_X(x) = \sum_y p(x, y)$$

$$p_Y(y) = \sum_x p(x, y)$$

  - Conditional pmf:

$$p_{X|Y}(x \mid y) = p\{X = x \mid Y = y\} = p(x, y) \big/ p_Y(y)$$

Jointly Distributed *r.v.s* (3)

- Probability density function (pdf) for continuous *r.v.s*
  - Joint:
    $$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$
    $$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\,du\,dv$$
  - Marginal:
    $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
    $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
  - Conditional:
    $$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

Hands-on Problem (1)

- Suppose two discrete *r.v.s* X and Y have the following joint *p.m.f* as shown in the table. Thus,

X assumes values 1, 2, and 3;

Y assumes values of 2, 3, and 4.

X \ Y	2	3	4
1	1/6	1/6	1/6
2	0	1/6	1/6
3	0	0	1/6

- Find the marginal *p.m.f.* p_X and p_Y
- Find the conditional *p.m.f.* of Y, given that X=1
- Calculate E[X] and Var[X]
- Find *p.m.f* for Z=X+Y

Jointly Distributed *r.v.s* (Agenda)

- Joint distribution function
- Marginal distribution function
- Joint/marginal/conditional *p.m.f.*
- Joint/marginal/conditional *p.d.f.*
- **Independency**
- Important parameters
- Two important properties

Related reading: Allen's Ch. 2.7

Jointly Distributed *r.v.s* (4)

- Independent r.v.s

Two *r.v.*s X and Y are independent if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\forall x\forall y$$

Discrete: $p(x,y) = p_X(x)p_Y(y)\forall x\forall y$

Continuous: $f(x,y) = f_X(x)f_Y(y)\forall x\forall y$

Hands-on Problem (1: Cont'd)

- Suppose two discrete *r.v.*s X and Y have the following joint *p.m.f* as shown in the table. Thus, X assumes values 1, 2, and 3; Y assumes values of 2, 3, and 4.

X \ Y	2	3	4
1	1/6	1/6	1/6
2	0	1/6	1/6
3	0	0	1/6

- ✓ Find the marginal *p.m.f.* p_X and p_Y:
- ✓ Find the conditional *p.m.f.* of Y, given that X=1:
- ✓ Calculate E[X] and Var[X]
- ✓ Find p.m.f for Z=X+Y
- **Are X and Y independent random variables? Why?**

Jointly Distributed *r.v.s (Agenda)*

- ✓ Joint distribution function
- ✓ Marginal distribution function
- ✓ Joint/marginal/conditional *p.m.f.*
- ✓ Joint/marginal/conditional *p.d.f.*
- ✓ Independency
- **Important parameters**
  - **Expectation, Variance, Covariance, Correlation**
- Two important properties

Jointly Distributed *r.v.s (5)*

- Expectation

X, Y are jointly distributed *r.v.*s, *g* is a function of 2 *r.v.*s

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\,p(x,y) \quad \text{for discrete}$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dxdy \quad \text{for continuous}$$

Properties (*Theorem 2.7.1*)

$$\begin{aligned}
&\boldsymbol{E[c] = c}\\
&\boldsymbol{E[cX] = cE[X]}\\
&\boldsymbol{E[X+Y] = E[X] + E[Y]}\\
&\boldsymbol{E[XY] = E[X]E[Y]} \quad \textbf{if} \quad \boldsymbol{X \amalg Y}\\
&\boldsymbol{E[g(X)h(Y)] = E[g(X)]E[h(Y)]} \quad \textbf{if} \quad \boldsymbol{X \amalg Y}
\end{aligned}$$

Jointly Distributed *r.v.s (6)*

Var(X)= E[(X-μ)^2]

- Covariance of X and Y, cov[X,Y]

$$\begin{aligned}
\text{cov}[X,Y] &= E[(X - E[X])(Y - E[Y])]\\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

Remarks

– If cov[X,Y]=0, X and Y are said to be *uncorrelated*

– Two independent r.v.s are uncorrelated, but not all uncorrelated r.v.s are independent!

Jointly Distributed *r.v.s (7)*

- Variance (*Theorem* 2.7.2)

  X, Y are *r.v.*s, c is a constant

$$\begin{aligned}
&Var[c] = 0 \\
&Var[cX] = c^2 Var[X] \\
&Var[X+Y] = Var[X] + Var[Y] + 2\,\mathrm{cov}[X,Y] \\
&Var[X+Y] = Var[X] + Var[Y] \quad \text{if} \quad X \coprod Y \\
&Var[X] = E[X^2] - E^2[X]
\end{aligned}$$

Jointly Distributed *r.v.s (8)*

- Correlation (coefficient) of X, Y, ρ (X,Y)

$$\rho[X,Y] = \frac{\mathrm{cov}[X,Y]}{\sqrt{Var[X]Var[Y]}},$$
$$Var[X] \neq 0 \quad \textbf{and} \quad Var[Y] \neq 0$$

Concepts and properties for two r.v.s can be extended to any finite number of r.v.s

Jointly Distributed *r.v.s (Agenda)*

- ✓ Joint distribution function
- ✓ Marginal distribution function
- ✓ Joint/marginal/conditional *p.m.f.*
- ✓ Joint/marginal/conditional *p.d.f.*
- ✓ Independency
- ✓ Important parameters
- **Two important properties: Max and Min**

Max Theorem 2.7.3

- Let $X_1, X_2, \ldots, X_n$ be n independent *r.v.s* with distribution function $F_{X1}, F_{X2}, \ldots, F_{Xn}$. Let *r.v.* $Y = g(X_1, X_2, \ldots, X_n)$, if $Y(\omega) = \boldsymbol{max}\{X_1(\omega), X_2(\omega), \ldots, X_n(\omega)\}$, $\omega \in \Omega$, then

$$\boldsymbol{F_Y(y) = F_{X1}(y) F_{X2}(Y) \ldots F_{Xn}(y) \quad \forall y \in R}$$

Hands-On Problem (2)

- An on-line airline reservation system uses 2 identical duplexed computer systems, each of which has an exponential time-to-failure with a mean of 2000 hours. Each CS has built-in redundancy so failures are rare. The entire CS fails only if both computers fail.
- Question: probability that the system will not fail during 1 week of continuous operation?

Min Theorem 2.7.4

- Let $X_1, X_2, \ldots, X_n$ be n independent *r.v.s* with distribution function $F_{X1}, F_{X2}, \ldots, F_{Xn}$. Let *r.v.* $Y = g(X_1, X_2, \ldots, X_n)$, if $Y(\omega) =$ ***min***$\{X_1(\omega), X_2(\omega), \ldots, X_n(\omega)\}$, $\omega \in \Omega$, then

$$F_Y(y) = 1-(1-F_{X1}(y))(1-F_{X2}(Y))\ldots(1-F_{Xn}(y)) \quad \forall y \in R$$

Hands-On Problem (3)

- A computer system (CS) consists 4 subsystems, each of which has the same exponential time-to-failure with a mean of 2000 hours. Each subsystem is independent but the entire CS fails if any of the subsystems fails.
- Question:
  - The mean time to entire system failure?
  - The probability that time-to-failure exceed 100 hours?

Hands-On Problem (4)

- A computing system consists of 3 processing units A, B, and C, each of which has the same exponential time-to-failure with parameter $\lambda=10^{-4}$/hour. Each unit is independent. The entire system functions correctly if both "A and B are working" or "C is working". In other words, the entire system fails if "either A or B fails" and "C fails" at the same time.
- Question:
  - The probability that the system will not fail during 10^4 hours of continuous operation?

Next Topics

- Statistical Inference
- Stochastic Processes

Things to Do

- Homework
- Project proposal due **February 23, Friday**