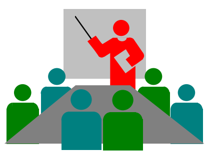
ECE560: Computer Systems Performance Evaluation



Lecture #15:
Queueing Networks

Instructor: Dr. Liudong Xing

Administration Issues (4/1/2024)

- Homework #5 due **Today**
- Homework #6 assigned
  - Due **April 8, Monday**
- Project final report
  - Due: **April 19, Friday**
  - Refer to Report Guidelines
- Today's topics
  - Finish L#14 (Priority Q Systems)
  - Then L#15 (Q Networks)

Review of Queueing Systems

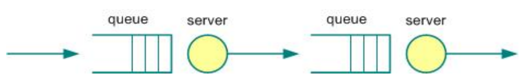
- Birth-and-death queueing systems (Lecture #11, 13)
  - Fit the birth-and-death process
  - M/M/ Queues: M/M/1, M/M/1/N, M/M/c, M/M/ ∞, M/M/1/k/k
  - Solution: Balance equations
- Embedded Markov-chain queueing systems (Lecture #14)
  - More general arrival process or service times are allowed: M/G/1, M/D/1, GI/M/1
  - Solution
    - constructing an embedded Markov chain
    - And applying Z-transform and Laplace-Stieltjes transform methods
  - $\pi_n = p_n$ *iff* the arrival process is Poisson
    - π_n: from "an arriving customer" point of view
    - p_n: from "a random observer" point of view
- Priority queueing systems (Lecture #14)
  - Some customers get preferential treatment
  - M/G/1, M/M/c

Topic

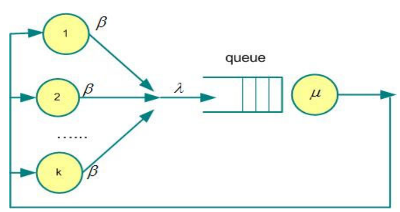
- **Concepts and notations**
- Product-form queueing networks

Queueing Networks (QN)

- A QN is a collection of simple queueing systems interconnected by directed links



- In general, a model in which jobs departing from one queue arrive at another queue (possibly the same queue) is called a QN
  - M/M/1/K/K is actually a QN

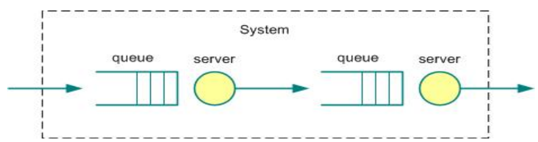


Classification of QN

- In terms of whether or not the QN accepts customers from outside the system
  - Open QN,
  - Closed QN,
  - Mixed QN

Open QN

- Customers enter from outside the system, circulating among the service centers for service, and depart from the system
- Having external arrivals and departures



New customers enter and
Old customers eventually leave

- Number of customers/jobs in the system varies with time
- Usually assuming the throughput γ is known (to be equal to the arrival rate), the goal is <u>to characterize the distribution of number of jobs in the system</u>

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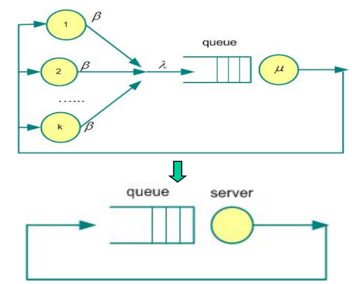
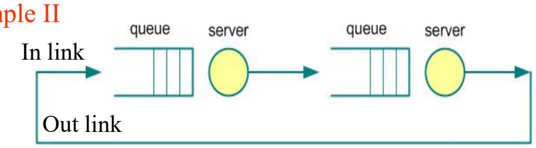
Closed QN

- A fixed number of customers circulate indefinitely among the service centers
- Having no external arrivals or departures

Example I:
M/M/1/k/k
(Machine repair model)



Example II



- Total number of customers/jobs in the system stays constant; a fixed number of customers contend for resources
- Usually assuming the number of jobs is given, the goal is <u>to determine the throughput (i.e., the job completion rate)</u>

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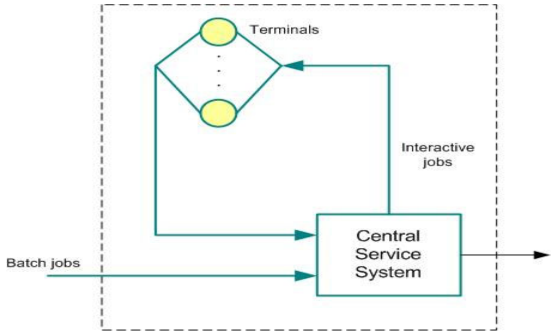
Mixed QN

- Open for some customers: who enter from the outside the system and eventually leave
- Closed for others: who always remain inside the system



With more than one class/type of customers -- multi-class systems

Customers within a class are indistinguishable (have the same service demand)

Notations

- K: the number of service centers in a QN
- C: number of customer classes
- S_{ck}: average service time per visit that the service center k provides for customers of class c
  - A customer may visit a service center several times to complete the service
- V_{ck}: average number of visits that a customer of class c makes to service center k
- Thus, the total service demand for customers of class c at service center k:

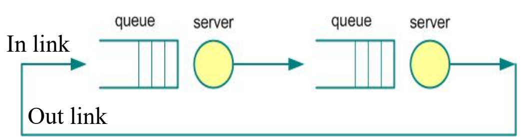
$$D_{ck} = V_{ck} * S_{ck}$$

- Thus, the total service demand at service center k:

$$D_k = \sum_c D_{ck}$$

Notations (Cont'd)

- γ: the average throughput of the system (Recall: γ is the rate at which jobs successfully depart from the system)
  - For open QN
    - number of jobs leaving the system per unit time defines γ
  - For closed QN (no jobs actually leaves the system)
    - Traversing the outside link (the link connecting Out to In) is equivalent to leaving the system and immediately reentering it
    - γ is defined as the number of jobs traversing this link per unit time.



- γ_{ck}: the average throughput of class c at the service center k

Forced Flow Law

- It's shown that:

$$\gamma = \frac{\gamma_{ck}}{V_{ck}}$$

  - For a single-class system:

$$\gamma = \frac{\gamma_k}{V_k} \quad \text{or} \quad \gamma_k = \gamma V_k$$

  - Called "the forced flow law"
    - Showing that the throughput of any service center determines the throughput of all the others
    - Relating the system throughput to individual device throughputs

- Example:

The measurements that a performance analyst made on his main batch processing systems show that the average number of visits each job makes to Drive 1 is 5; and that the disk throughput for Drive 1 is 10 requests per second. Find the system throughput γ.

The Bottleneck

- Saturated device
  - a device (server) with a utilization of 100%
- A system is saturated when at least one of its servers or resources is saturated
- The bottleneck of the system
  - The first device (server) to saturate as the load on the system is increased
- Identification based on service demand D_k, $k=1,2, \ldots, K$:

  The bottleneck is device *j* if

$$D_j = D_{\max} = \max\{D_1, \ldots, D_K\}$$

The device with the highest service demand has the highest utilization and is the bottleneck device.

The Bottleneck (Cont'd)

- The bottleneck is **workload dependent**
  - Different workloads may have different bottlenecks for the same computer systems
    - Scientific computing jobs tend to be CPU bound
    - Business-oriented jobs (E-mail, DBMS) tend to be I/O bound
  - The workload on computer systems usually varies during different period of the day, so do the bottleneck
  - Most interested in the bottleneck during the peak period of the day

The Bottleneck (Cont'd)

- Example

  A performance analyst measures a small batch processing computer system. He finds that the CPU (numbered by 1) has a visit ratio of 20 (i.e., V1=20) with S1=0.05 sec, the first I/O device has V2=11 with S2=0.08 sec, while the second I/O device has V3=8 with S3=0.04 sec. What is the bottleneck of the system, CPU, or the first I/O device or the second I/O?

Topic

- ✓ Concepts and notations
- **Product-form queueing networks**
  - Concept
  - Case studies
  - Properties

Product-Form QN

- In general, any QN in which the expression for joint equilibrium (steady-state) probability has the form of

$$\pi(n_1, n_2, ..., n_K) = \frac{1}{G(N)} \prod_{i=1}^{K} f_i(n_i)$$

  - $\pi(n_1,n_2,...,n_K)$: probability of n_1 customers in center 1, n_2 customers in center 2, …, n_K customers in center K.
  - K: total number of nodes in the QN
  - n_i: number of customers at the i*th* service facility
  - $f_i(n_i)$: some function of n_i
  - N: total number of jobs in the system
  - $G(N)$: a normalizing constant
- Also called separable QN

We focus on Product-Form QN!

Case Studies

- Two open product-form QN
  - A series of K M/M/1 queues
  - Jackson QN
- A closed 3-stage product-form QN

Case Study I: Open PF-QN

- Consider a series of K M/M/1 queues

- Jobs leaving a queue immediately join the next queue
- Each individual queue in the series can be analyzed independently
  - Arrival rate: λ
  - Service rate of the i-th server: μ_i
  - Thus,

utilization of the i-th server

$$\rho_i = \frac{\lambda}{\mu_i}$$

probability of n_i jobs in the i-th queue system (including queue and server):

$$\pi_i(n_i) = (1 - \rho_i)\rho_i^{n_i}$$

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Case Study I (Cont'd)

– The joint probability of K queues $\pi(n_1, n_2, \ldots, n_K)$:

$$\begin{aligned}
&\pi(n_1, n_2, \ldots, n_K) \\
&= (1-\rho_1)\rho_1^{n_1} \times (1-\rho_2)\rho_2^{n_2} \times \ldots \times (1-\rho_K)\rho_K^{n_K} \\
&= \pi_1(n_1)\pi_2(n_2) .. \pi_K(n_K)
\end{aligned}$$

It is a product-form QN.

$$\pi(n_1, n_2, \ldots, n_K) = \frac{1}{G(N)} \prod_{i=1}^{K} f_i(n_i)$$

Dr. Xing QN(I) 20

Case Studies (Agenda)

- Two open product-form QN
  - ✓ A series of K M/M/1 queues
  - **Jackson QN**
    - **Jackson theorem and application**
    - 3-step Jackson algorithm
    - An example
- A closed 3-stage product-form QN

Case Study II: **Jackson QN**

- A Jackson QN contains K nodes satisfying 3 properties:
  - Each node k consists of c_k identical exponential servers, each with service rate μ_k
  - Customers arriving at node k from outside the system arrive in a Poisson pattern with average arrival rate λ_k. Customers also arrive at node k from other nodes within the QN.
  - Once served at node k, a customer immediately goes to node j (j=1,…, K) with probability p_{kj}; or leaves the QN with probability

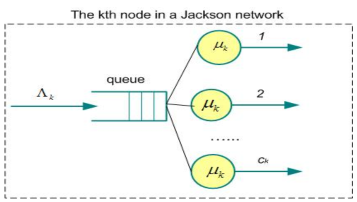
$$1-\sum_{j=1}^{K} p_{kj}$$

Jackson's Theorem 6.2.1

- The average arrival rate to each node k:

$$\Lambda_k = \lambda_k + \sum_{j=1}^{K} \Lambda_j p_{jk}$$

- Each node k behaves like an independent M/M/c_k queueing system with the average arrival rate Λ_k and average service rate μ_k for each of the c_k servers



- The steady state probability that there are n_k customers in the k-th node for k=1,2,…, K:

$$\pi(n_1, n_2, \ldots, n_K) = \pi_1(n_1)\pi_2(n_2)..\pi_K(n_K)$$

given

$$\Lambda_k < c_k \mu_k$$

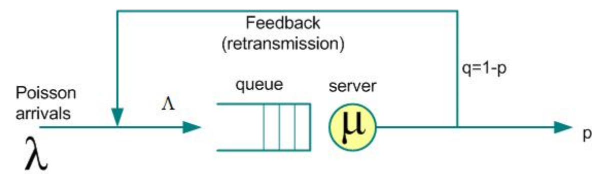
  – $\pi_k(n_k)$ is the steady-state probability that there are n_k customers in the k-th node if treated as an M/M/c_k with Λ_k and μ_k for each of the c_k servers

Jackson QN (Cont'd)

- The QN in the case study I is a Jackson QN

- Application – M/M/1 queue with feedback

A message switching center represented by a M/M/1 queue transmits the message to the required destination. Assume the service (time to transmit a message and receive an ack of correct receipt) is exponential. Assume an error detecting code is used. The probability that a msg is received correctly is p; with probability $q=1-p$ the msg must be retransmitted. Find ρ, L, W.



Note on Jackson Networks

- There is only one job class in the network
- The overall number of jobs in the network is unlimited
- Each of the K nodes in the network can have arrivals from outside (λ_k)
- A job can leave the network from any node
- All service times and inter-arrival times are exponentially distributed
- The service discipline at all nodes is FCFS.

Case Studies (Agenda)

- Two open product-form QN
  - ✓ A series of K M/M/1 queues
  - ✓ Jackson QN
    - ✓ Jackson theorem and application
    - **3-step Jackson algorithm**
    - An example
- A closed 3-stage product-form QN

Jackson Networks Algorithm

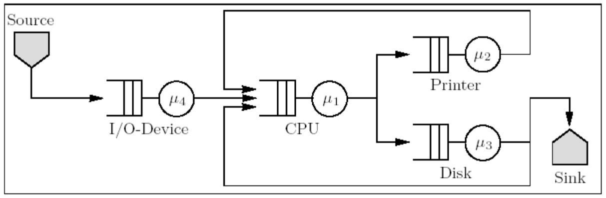
- Based on Jackson Theorem
  - **Step 1**: For all nodes, i=1,…,K, compute the average arrival rates Λ_i of the network by solving the **traffic equations**:

$$\Lambda_k = \lambda_k + \sum_{j=1}^{K} \Lambda_j p_{jk}$$

  - **Step 2**: Consider each node as an M/M/c_k queueing system. Check the stability (if ρ_k<1), and compute the marginal state probabilities $\pi_k(n_k)$, and performance measures (L,W, L_q, W_q) of each node using the formulae for M/M/c_k systems
  - **Step 3**: Computer the (joint) steady-state probabilities of the overall queueing network.

$$\pi(n_1, n_2, \ldots, n_K) = \pi_1(n_1)\pi_2(n_2)..\pi_K(n_K)$$

Hands-on Problem



- Number of nodes K=4
- Service times exponentially distributed with:

$$1/\mu_1 = 0.04,\ 1/\mu_2 = 0.03,\ 1/\mu_3 = 0.06,\ 1/\mu_4 = 0.05$$

- Inter-arrival times exponentially distributed with λ_4 = 4 jobs/sec
- Scheduling discipline: FCFS
- Transition probabilities (0 – source/sink):

$$p_{12} = p_{13} = 0.5, \quad p_{41} = p_{21} = 1, \quad p_{31} = 0.6, \quad p_{30} = 0.4$$

Hands-on Problem (Cont'd)

- Find the arrival rates to each node?
- Find the following performance measures:
  - Utilization of each node (1-4)
  - Mean number of jobs in each node
  - Mean response times of each node
  - Mean overall response time
  - Mean waiting time in the queue of each node
  - Mean queue length for each node
  - Marginal steady-state probabilities: for example, find $\pi_1(3)$, $\pi_2(2)$, $\pi_3(4)$, $\pi_4(1)$?
- Find the steady-state joint probabilities, for example, $\pi(3,2,4,1)$?

Reference

- Performance measures of the M/M/1 queue

$$\alpha = \rho = \lambda / \mu$$

$$\pi_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho,\ \pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0 = \rho^n (1-\rho)$$

$$\begin{aligned}\gamma &= \mu P[> 0 \text{ jobs in the system}] \\ &= \mu(1 - P[0 \text{ jobs in the system}]) \\ &= \mu(1-\pi_0) = \mu(1-(1-\rho)) = \mu\rho = \lambda\end{aligned}$$

$$\begin{aligned}L &= \sum_{n=0}^{\infty} n\pi_n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n \\ &= (1-\rho)\rho\sum_{n=1}^{\infty} n\rho^{n-1} = \frac{(1-\rho)\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}\end{aligned}$$

$$W = L/\lambda = \frac{\rho}{1-\rho}/\lambda = \frac{1}{\mu-\lambda}$$

$$\begin{aligned}L_q &= L - (1 * P[\text{Server is not empty}]) \\ &= L - (1 - P[0 \text{ jobs in the system}]) \\ &= L - (1-\pi_0) = L - (1-(1-\rho)) \\ &= L - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho}\end{aligned}$$

$$W_q = L_q/\lambda = \frac{\rho^2}{1-\rho}\frac{1}{\lambda} \text{ or } W_q = W - W_s = \frac{\rho}{(1-\rho)\lambda} - \frac{1}{\mu} = \frac{\rho^2}{1-\rho}\frac{1}{\lambda}$$

Case Studies (Agenda)

- Two open product-form QN
  - ✓ A series of K M/M/1 queues
  - ✓ Jackson QN
    - Jackson theorem and application
    - 3-step Jackson algorithm
    - A computer system example

$$\Lambda_k = \lambda_k + \sum_{j=1}^{K} \Lambda_j p_{jk}$$

$\pi_k(n_k)$, and performance measures (L, W, L_q, W_q) of each node

$$\pi(n_1, n_2, \ldots, n_K) = \pi_1(n_1)\pi_2(n_2)\ldots\pi_K(n_K)$$

- **A closed 3-stage product-form QN**

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Review

- Closed QN
  - – Having no external arrivals or departures
  - – A fixed number of customers circulate indefinitely among the service centers
  - – Total number of customers/jobs in the system stays constant; a fixed number of customers contend for resources

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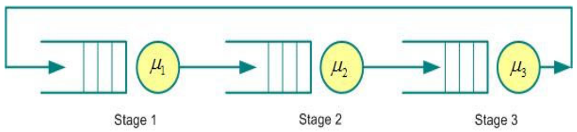
Case Study III: An Example Closed PF-QN

- Consider a closed 3-stage QN



- There are two customers
- Jobs leaving a queue immediately join the next queue
- Each queue in the network has exponential service time with a mean of $1/\mu_i$ (i=1,2,3)
- Can be described as a Markov process with each state described as the triplet

$$S = \{n_1, n_2, n_3\} \text{ where } \sum_{i=1}^{3} n_i = 2$$

- n_i: number of customers in the queue system i
- The state transition diagram (extra note)
- The steady-state solution:
  - Balance equations
  - Sum of all state probabilities equal 1

Closed PF-QN (Cont'd)

- Balance equations:

State	Rate out=Rate in
1: (2,0,0)	$\pi(2,0,0)\mu_1 = \pi(1,0,1)\mu_3$
2: (1,1,0)	$\pi(1,1,0)(\mu_1+\mu_2) = \pi(2,0,0)\mu_1 + \pi(0,1,1)\mu_3$
3: (1,0,1)	$\pi(1,0,1)(\mu_1+\mu_3) = \pi(1,1,0)\mu_2 + \pi(0,0,2)\mu_3$
4: (0,2,0)	$\pi(0,2,0)\mu_2 = \pi(1,1,0)\mu_1$
5: (0,1,1)	$\pi(0,1,1)(\mu_2+\mu_3) = \pi(0,2,0)\mu_2 + \pi(1,0,1)\mu_1$
6: (0,0,2)	$\pi(0,0,2)\mu_3 = \pi(0,1,1)\mu_2$

- The sum of all state probabilities equal 1

$$\sum_{i=1}^{6} \pi(S_i) = 1$$

Closed PF-QN (Cont'd)

- Steady-state solution has the form of

$$\pi(n_1,n_2,n_3) = G(1/\mu_1)^{n_1}(1/\mu_2)^{n_2}(1/\mu_3)^{n_3}$$

  - *G*: a normalization constant to ensure all probabilities sum to 1

$$G = \frac{1}{\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{2}\pi(i,j,k)}$$

- Expected # of customers in each stage:

$$L_i = E[N_i] = \sum_{j=0}^{2} jP[N_i = j]$$

$$\text{Marginal probability}: P[N_1 = j] = \sum_m \sum_k \pi(j,k,m)$$

- Expected time spent in each stage:

$$\lambda_1 = \mu_3 \Pr\{\text{server 3 is busy}\} = \mu_3(1 - P[N_3 = 0])$$
$$\lambda_2 = \mu_1 \Pr\{\text{server 1 is busy}\} = \mu_1(1 - P[N_1 = 0])$$
$$\lambda_3 = \mu_2 \Pr\{\text{server 2 is busy}\} = \mu_2(1 - P[N_2 = 0])$$
$$W_i = L_i / \lambda_i$$

Agenda

✓ Concepts and notations

✓ Product-form queueing networks

- ✓ Concept
- ✓ Case studies
  - ✓ Open PF-QN
  - ✓ Closed PF-QN
- **Important properties of product-form QN**
  - Local balance
  - M → M property

PF-QN Property (1)

- Global balance vs. Local balance
  - The equations: (rate in=rate out)

$$\forall i \in S: \quad \sum_{j \in S} \pi_j q_{ji} = \pi_i \sum_{j \in S} q_{ij}$$

with the normalizing condition: $\sum_{i \in S} \pi_i = 1$
are called the

"*Global Balance Equations*"

  - Global balance:

    The transition rate out of a state of a QN = transition rate into this state of the QN

  - For so-called product-form QN exists also a *local balance*:

    The transition rate out of a state of a QN due to a departure from node *i* = transition rate into this state of this QN due to an arrival to node *i*.

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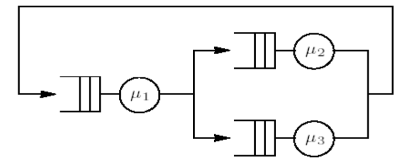
Example

- Consider a closed QN with 3 nodes



  - Number of customers m=2
  - Service time exponentially distributed with:

$$\mu_1 = 4/\text{sec},\ \mu_2 = 1/\text{sec und } \mu_3 = 2/\text{sec}$$

  - Scheduling discipline: FCFS
  - Transition probabilities:

$$p_{12} = 0.4,\ p_{13} = 0.6,\ p_{21} = p_{31} = 1$$

  - State space of the MC:

$$\{(2,0,0),\ (0,2,0),\ (0,0,2),\ (1,1,0),\ (1,0,1),\ (0,1,1)\}$$

where, state $S=(n_1,n_2,n_3)$: n_1 jobs in node 1, n_2 jobs in node 2 and n_3 jobs in node 3

Find the steady-state probabilities $\pi(n_1, n_2, n_3)$

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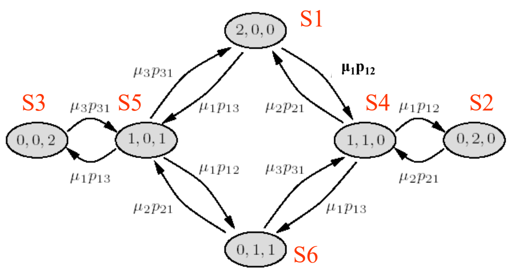
State Transition Diagram



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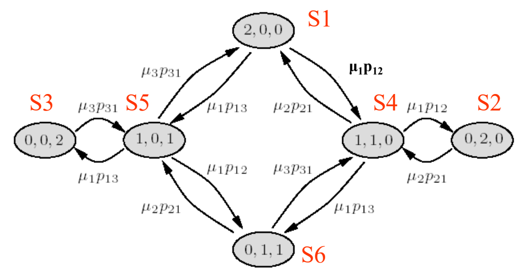
Global Balance Equations

- State transition diagram



Rate Out = Rate In

$$
\begin{aligned}
(1)\quad & \pi(2,0,0)(\mu_1 p_{12} + \mu_1 p_{13}) = \pi(1,0,1)\mu_3 p_{31} + \pi(1,1,0)\mu_2 p_{21}\,, \\
(2)\quad & \pi(0,2,0)\mu_2 p_{21} = \pi(1,1,0)\mu_1 p_{12}\,, \\
(3)\quad & \pi(0,0,2)\mu_3 p_{31} = \pi(1,0,1)\mu_1 p_{13}\,, \\
(4)\quad & \pi(1,1,0)(\mu_2 p_{21} + \mu_1 p_{13} + \mu_1 p_{12}) = \pi(0,2,0)\mu_2 p_{21} + \pi(2,0,0)\mu_1 p_{12} \\
& \qquad + \pi(0,1,1)\mu_3 p_{31}\,, \\
(5)\quad & \pi(1,0,1)(\mu_3 p_{31} + \mu_1 p_{12} + \mu_1 p_{13}) = \pi(0,0,2)\mu_3 p_{31} + \pi(0,1,1)\mu_2 p_{21} \\
& \qquad + \pi(2,0,0)\mu_1 p_{13}\,, \\
(6)\quad & \pi(0,1,1)(\mu_3 p_{31} + \mu_2 p_{21}) = \pi(1,1,0)\mu_1 p_{13} + \pi(1,0,1)\mu_1 p_{12}\,.
\end{aligned}
$$

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Local Balance Equations

- State (1,1,0)
  - Node 2: transition rate out of state (1,1,0) due to a departure from node 2 = transition rate into this state due to an arrival to node 2.

$$(4') \quad \pi(1,1,0)\cdot\mu_2\cdot p_{21} = \pi(2,0,0)\cdot\mu_1\cdot p_{12}$$

  - Node 1

$$(4'') \quad \pi(1,1,0)\cdot\mu_1\cdot(p_{13}+p_{12}) = \pi(0,1,1)\cdot\mu_3\cdot p_{31} + \pi(0,2,0)\cdot\mu_2\cdot p_{21}$$

  - By adding these local balance equations, (4`) and (4``), we get the global balance equation (4) for state (1,1,0).

- The equations (1), (2), (3) are already local balance equations!
  - State (2,0,0), node 1 → Eq. (1)
  - State (0,2,0), node 2 → Eq. (2)
  - State (0,0,2), node 3 → Eq. (3)

Dr. Xing QN(I) 41

Local Balance Equations (Cont'd)

- The equations (5) and (6) can be split to local balance equations similar to equation (4)

$$(5') \quad \pi(1,0,1)\mu_1(p_{12}+p_{13}) = \pi(0,1,1)\mu_2 p_{21} + \pi(0,0,2)\mu_3 p_{31}\,,$$
$$(5'') \quad \pi(1,0,1)\mu_3 p_{31} = \pi(2,0,0)\mu_1 p_{13}\,,$$
$$(6') \quad \pi(0,1,1)\mu_2 p_{21} = \pi(1,0,1)\mu_1 p_{12}\,,$$
$$(6'') \quad \pi(0,1,1)\mu_3 p_{31} = \pi(1,1,0)\mu_1 p_{13}\,,$$

$$(5') + (5'') = (5) \text{ und } (6') + (6'') = (6)$$

- (5`): state (1,0,1), node 1
- (5``): state (1,0,1), node 3
- (6`): state (0,1,1), node 2
- (6``): state (0,1,1), node 3

Dr. Xing QN(I) 42

Solution

- Steady state probabilities

$$\pi(1,0,1) = \pi(2,0,0)\frac{\mu_1}{\mu_3}p_{13}\,, \qquad \pi(1,1,0) = \pi(2,0,0)\frac{\mu_1}{\mu_2}p_{12}\,,$$

$$\pi(0,0,2) = \pi(2,0,0)\left(\frac{\mu_1}{\mu_3}p_{13}\right)^2, \qquad \pi(0,2,0) = \pi(2,0,0)\left(\frac{\mu_1}{\mu_2}p_{12}\right)^2,$$

$$\pi(0,1,1) = \pi(2,0,0)\frac{\mu_1^2}{\mu_2\mu_3}p_{12}p_{13}\,.$$

- Normalizing condition leads to

$$\pi(2,0,0) = \left[1 + \mu_1\left(\frac{p_{13}}{\mu_3} + \frac{p_{12}}{\mu_2} + \frac{\mu_1 p_{13}^2}{\mu_3^2} + \frac{\mu_1 p_{12}^2}{\mu_2^2} + \frac{\mu_1 p_{12} p_{13}}{\mu_2\mu_3}\right)\right]^{-1}$$

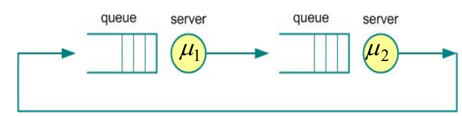
- Resulted steady-state probabilities:

$$\pi(2,0,0) = \underline{0.103}\,, \quad \pi(0,0,2) = \underline{0.148}\,, \quad \pi(1,0,1) = \underline{0.123}\,,$$
$$\pi(0,2,0) = \underline{0.263}\,, \quad \pi(1,1,0) = \underline{0.165}\,, \quad \pi(0,1,1) = \underline{0.198}\,.$$

- From these joint steady-state probabilities, the marginal probabilities and performance measures (L_i, W_i) can be calculated

Hands-On Problem

- Consider a closed QN with 2 nodes



  – Number of customers N=2
  – Service time exponentially distributed with:

$$\mu_1 = 4/\text{sec} \qquad \mu_2 = 3/\text{sec}$$

  – Scheduling discipline: FCFS
  – Transition probabilities:

$$p_{12} = 1 \qquad p_{21} = 1$$

1. Draw the state transition diagram for this system
2. Find the global balance equations for each state
3. Find the local balance equations for each state
4. Find the joint steady-state probabilities for all states
5. Find the mean number of customers in each stage/node
6. Find the average time a customer spent in each stage/node

Note!

- The local balance equations can be solved much easier than the global balance equations
  - The rate at which jobs enter a single node of the QN is equal to the rate at which they leaves it
  - Thus, local balance is concerned with a local situation and reduces the computational effort.
- The solution is still very complex for greater QN.

Product Form and Local Balance

- Local balance yields a solution with the product-form property.

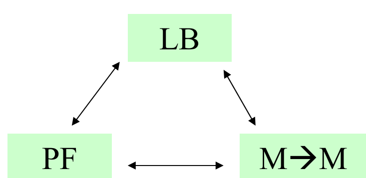
$$\pi(n_1, n_2, \ldots, n_K) = \frac{1}{G}\prod_{i=1}^{K} \pi_i(n_i)$$

  - The steady-state probability for the state (n_1, n_2, …, n_K) is the product of the marginal probabilities $\pi_i(n_i)$ for the single nodes
  - The normalizing constant G can be obtained from the normalizing condition (sum of all steady-state probabilities equals 1)
  - For QN with product-form solution, the local balance property holds.

A necessary and sufficient condition for the existence of product form solutions is given in the local-balance property.

PF-QN Property (2)

- M→ M property (Markov implies Markov)
  - A node (service center) has the M → M property *iff* the node transforms a Poisson arrival process into a Poisson departure process.
- Muntz has shown that a QN has a product form solution if all nodes of the network have the M → M property.
- If a node satisfies local balance, it must have M → M property



Next Topics

- Petri Nets Modeling

Things to Do

- Homework
- Class project