Department of Electrical and Computer Engineering
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ECE560: Computer Systems Performance Evaluation

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Homework #4 Solution

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Problem 1:

Solution:

a) $E[T] = \frac{1}{\lambda} = \frac{1}{0.01}$ hours $= 100$ hours

b) $P_n(t) = P[Y_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$

Let $t = 1$ day $= 24$ hours, $n = 1$ failure

$$P[Y_t = 1] = e^{-\lambda \cdot 24} \frac{(\lambda \cdot 24)^1}{1!} = e^{-0.01 \times 24} \times (0.01 \times 24)^1$$

$$= 0.1888$$

c) $P[Y_t < 3] = \sum_{k=0}^{2} P[Y_t = k] = \sum_{k=0}^{2} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$

$\lambda = 0.01/\text{hr}$ $t = 1$ week $= 168$ hrs

$$P[Y_t < 3] = \sum_{k=0}^{2} e^{-0.01 \times 168} \frac{(0.01 \times 168)^k}{k!} = 0.7625$$

d). $P[Y_t = 0] = e^{-\lambda t} \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$ for $t = 24$ hrs

$$= e^{-0.01 \times 24} = 0.7866$$

e). Since the time between failurs is exponentially distributed, by memory-less property, we have:

$$E[T] = \frac{1}{\lambda} = 100 \text{ hours}$$

f) According to the "Decomposition Property of a Poisson process", the terminal failure can be described by a Poisson process with rate of

$$\lambda_{terminal} = P_{terminal} * \lambda = \frac{3}{5} \times 0.01 = 0.006 \,/\text{hr}$$

And mean time between terminal failures is:

$$1/\lambda_{terminal} = \frac{1}{0.006} \text{ hours} = 167 \text{ hours}$$

Probability of 1 terminal failure in a 1-day period:

$$e^{-\lambda_{terminal} t} \frac{(\lambda_{terminal} \cdot t)^k}{k!}$$ where $k=1$, $\lambda_{terminal} = 0.006/hr$, $t = 1\text{day} = 24$ hours

$$= 0.1247$$

Problem 2: (part c is for your reference)

(5) a)
$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ 0 & a & 1-a \end{bmatrix}$$

(10) b) $\pi = \pi \cdot P \Rightarrow$

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ 0 & a & 1-a \end{bmatrix} \Rightarrow$$

$$\pi_1 = 0 \times \pi_1 + 1 \cdot \pi_2 + 0 \cdot \pi_3 = \pi_2$$
$$\pi_2 = (1-p)\pi_1 + 0 \cdot \pi_2 + a \cdot \pi_3 = (1-p)\pi_1 + a \cdot \pi_3$$
$$\pi_3 = p \cdot \pi_1 + 0 \cdot \pi_2 + (1-a)\pi_3 = p \cdot \pi_1 + (1-a)\pi_3$$

Solving two of the above equations, together with $\pi_1 + \pi_2 + \pi_3 = 1$, we get:

$$\pi_1 = \pi_2 = \frac{a}{p+2a}$$
$$\pi_3 = \frac{p}{p+2a} \qquad \therefore \pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{a}{p+2a}, \frac{a}{p+2a}, \frac{p}{p+2a}\right)$$

(5) c) By Eq. (4.67), or Lecture #10

$$m_2 = \frac{1}{\pi_2} = \frac{p+2a}{a} = \frac{p}{a} + 2$$

Problem 3:

M/M/1 Queue Model:

$\lambda = 4$ customers/minute

$W = 4$ minutes

By "Little's Law", we have

$$L = \lambda \cdot W = \boxed{16 \text{ customers}}$$

Problem 4:

M/M/1 queue model :

$\lambda = 15$ /hour $= \frac{15}{60}$ /min $= \frac{1}{4}$ /min

$W_s = 2.5$ min $\Rightarrow \mu = \frac{1}{W_s} = 0.4$ /min

(a) $\rho = \lambda/\mu = \frac{1}{4}/0.4 = 0.625$

$L_q = \frac{\rho^2}{1-\rho} = \frac{0.625^2}{1-0.625} = \boxed{1.0417}$ customers

<b>. By "Little's Law" :

$W_q = L_q/\lambda = 1.0417 \times 4 = \boxed{4.17}$ minutes

<c> $W = W_q + W_s = 4.17 + 2.5 = \boxed{6.67}$ minutes

<d> By "Little's Law"

$L = W \cdot \lambda = 6.67/4 = \boxed{1.67}$ customers

<e> P_r[Mirthless is idle] $= P_r$[0 customer in the system]

$= \pi_0 = 1 - \lambda/\mu = 1 - \rho = \boxed{0.375}$

<f> P_r[wait] $= P_r$[2 or more customers in the system]

$= \sum_{n=2}^{\infty} \pi_n = \sum_{n=2}^{\infty} \rho^n \cdot \pi_0 = \pi_0 \left[\sum_{n=0}^{\infty} \rho^n - \rho^0 - \rho^1\right]$

$= \pi_0 \left[\frac{1}{1-\rho} - 1 - \rho\right] = 0.375 \left[\frac{1}{1-0.625} - 1 - 0.625\right]$

$= \boxed{0.390625}$