ECE560: Computer Systems Performance Evaluation
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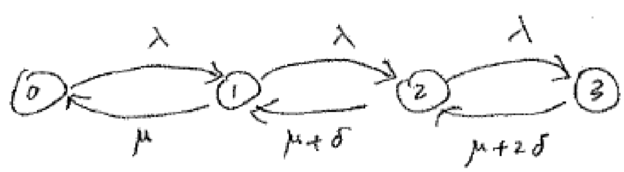
Homework #5 Solution

Problems:

1. Consider a computer system with one processor and a queue with 2 buffers. The job requests arrive to the processor at the rate of 16 requests per second with Poisson pattern. The time to service a job request at the processor is exponentially distributed with a mean of 50 milliseconds. Assume a job request in the queue and not being serviced can depart without service; this behavior is called “defect”. Assume the defect process is also exponential with the constant rate of $\delta = 2$ requests/second.
   a. Draw the complete state-transition diagram.
   b. What is the probability of the entire system is idle?
   c. What is the effective arrival rate of job requests into the system?
   d. What is the average number of job requests in the system?
   e. What is the average response time of a job?
   f. What is the average waiting time in the queue of a job?
   g. What is the average number of job requests in the queue?

(a)



(b)

$\lambda = 16/\text{sec}$

$w_s = 50\,\text{ms}$

$\mu = \frac{1}{w_s} = 20/\text{sec}$

$\delta = 2/\text{sec}$

state
0
1
2
3

rate in = rate out

$\pi_1 \cdot \mu = \lambda \cdot \pi_0$ ①

$\pi_0 \cdot \lambda + \pi_2 \cdot (\mu + \delta) = (\lambda + \mu) \cdot \pi_1$ ②

$\pi_1 \cdot \lambda + \pi_3 \cdot (\mu + 2\delta) = (\lambda + \mu + \delta) \cdot \pi_2$

$\pi_2 \cdot \lambda = (\mu + 2\delta) \cdot \pi_3$ ③

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ ④

By ①: $\pi_1 = \frac{\lambda}{\mu}\pi_0 = 0.8\pi_0$

① + ②: $\pi_2(\mu+\delta) = \lambda\pi_1 \Rightarrow \pi_2 = \frac{\lambda}{\mu+\delta}\pi_1 = \frac{\lambda^2}{\mu(\mu+\delta)}\pi_0 = 0.582\pi_0$

By ③: $\pi_3 = \frac{\lambda}{\mu+2\delta}\pi_2 = \frac{\lambda^3}{\mu(\mu+\delta)(\mu+2\delta)}\pi_0 = 0.388\pi_0$

By ④: $\pi_0 + \frac{\lambda}{\mu}\pi_0 + \frac{\lambda^2}{\mu(\mu+\delta)}\pi_0 + \frac{\lambda^3}{\mu(\mu+\delta)(\mu+2\delta)}\pi_0 = 1$

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu+\delta)} + \frac{\lambda^3}{\mu(\mu+\delta)(\mu+2\delta)}}$$

$$= \frac{1}{1 + 0.8 + \frac{16^2}{20*22} + \frac{16^3}{20*22*24}} = \frac{1}{1+0.8+0.582+0.388}$$

$$= \frac{1}{2.7697} = 0.361$$

ⓒ $\rho = 1-\pi_0 = \frac{\lambda_{eff}}{\mu}$

$$\lambda_{eff} = \mu(1-\pi_0)$$

$$= 20 * (1-0.361) = 12.779$$

(vs. 13.225 for M/M/1/3 without defect)

ⓓ $\pi_1 = \frac{\lambda}{\mu}\pi_0 = 0.8\pi_0 = 0.2888$

$\pi_2 = 0.582\pi_0 = 0.21$

$\pi_3 = 0.388\pi_0 = 0.14$

$$L = \sum_{n=0}^{3} n\pi_n = 0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 = 0.2888 + 2*0.21 + 3*0.14$$

$$= 1.1289$$

Extra questions :

Average response time : $W = \frac{L}{\lambda_{eff}} = \frac{1.1289}{12.779} = 0.0883$ sec $= 88.3$ ms

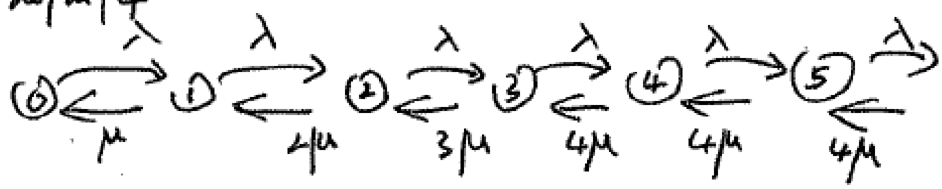
Average waiting time in the queue : $W_Q = W - W_S = 88.3 - 50 = 38.3$ ms

Average number of job requests in the queue :

$$L_Q = W_Q * \lambda_{eff} = 12.779 \times 0.0383 = 0.4894$$

2. A computer information center provides 4 consultants to help personal computer (PC) users solve their problems. PC users with problems arrive randomly with Poisson pattern, at an average rate of 40 per 8-hour day. The amount of time that a consultant spends with a PC user has an exponential distribution with mean value of 30 minutes. Users are assigned to the consultants in the order of their arrival. Assume the center can be modeled as a birth-and-death queuing system with infinite-capacity queue. Determine the following:
   a. Draw the state transition diagram (at lease show the first 6 states)
   b. The percentage of the time each consultant is busy, i.e., average consultant utilization
   c. Probability that all the consultants are idle.
   d. Probability that the information center is not idle.
   e. Probability that an arriving user has to wait for service
   f. The mean time a user spends in the information center
   g. Probability that all the consultants are busy and exactly 2 users are waiting in line

(a) M/M/4



(b) $\lambda = 40/8hr = \frac{40}{8 \times 60 \text{ min}} = \frac{1}{12}/\text{min}$

$W_s = 30 \text{ min} \Rightarrow \mu = \frac{1}{30}/\text{min}$

$\alpha = \lambda \cdot W_s = 30/12 = 2.5$

$\rho = \frac{\alpha}{c} = \frac{\alpha}{4} = \underline{\underline{0.625}}$

(c) $Pr\{\text{all 4 consultants are idle}\} = \pi_0$

$$= \left[\sum_{n=0}^{c-1} \frac{\alpha^n}{n!} + \frac{\alpha^c}{c!(1-\frac{\alpha}{c})} \right]^{-1} \qquad \alpha = 2.5 \quad c = 4$$

$$= \left[1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{\alpha^4}{24 \times 0.375} \right]^{-1} = (13.5694)^{-1}$$

$= 0.0737$

(d) $1 - \pi_0 = 1 - 0.0737 = 0.9263$

(e) $$Pr[\text{wait}] = C[c, \alpha] = \frac{\alpha^c/c!}{(1-\rho) \sum_{n=0}^{c-1} \frac{\alpha^n}{n!} + \frac{\alpha^c}{c!}}$$

$$= \frac{1.6276}{0.375 \left[1 + \frac{\alpha}{1} + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} \right] + \frac{\alpha^4}{4!}} = 0.3199$$

(f) $W = W_q + W_s \qquad W_s = 30$ min

$$W_q = C[c, \alpha] \cdot W_s \Big/ c(1-\rho)$$

$$= \frac{0.3199 * 30}{4 * 0.375} = 6.3972 \text{ mm.}$$

$W = 6.3972 + 30 = 36.3972$ min

(g) $\pi_6 = \frac{\alpha^6}{c!\, c^{6-c}} \pi_0 = 0.0469$

$$\begin{cases} \alpha = 2.5 \\ c = 4 \\ \pi_0 = 0.0737 \end{cases}$$