ECE560: Computer Systems Performance Evaluation
Homework #6 Solution

Problems:

1. Consider the GI/M/1 queueing systems, where the inter-arrival time τ has an Erlang-2 distribution with parameter λ (refer to Chapter 3.2.6 about the Erlang-*k* distribution, its LST is $A^*[\theta] = \left(\frac{k\lambda}{k\lambda+\theta}\right)^k$, here in the problem k=2), the service time is exponentially distributed with a mean of $1/\mu$.
   a) What is the steady-state probability that an arriving customer finds the system empty?
   b) If λ is 15 per hour and μ is 20 per hour, what is the probability that an arriving customer finds 4 customers in the system?
   c) What is the average number of customers in the system?
   d) What is the average waiting time of a customer in the queue of the system?

Note: The following derivation of LST is for your reference. You may use the LST given in the problem or textbook directly.

a) Step #1: find LST $A^*[\theta]$

By Eq (3.95) on P141, the p.d.f. of Erlang-2 with parameter λ is:

$$f(\tau) = \begin{cases} 2\lambda\cdot(2\lambda\tau)\, e^{-2\lambda\tau} & \tau > 0 \\ 0 & \tau \le 0 \end{cases}$$

$$\therefore A^*[\theta] = E[e^{-\theta\tau}] = \int_{-\infty}^{\infty} e^{-\theta\tau} f(\tau)\, d\tau$$

$$= \int_0^\infty (2\lambda)^2 \tau\, e^{-2\lambda\tau}\cdot e^{-\theta\tau}\, d\tau$$

$$= (2\lambda)^2 \int_0^\infty \tau\, e^{-(\theta+2\lambda)\tau}\, d\tau$$

$$= (2\lambda)^2 \int_0^\infty \tau\, \frac{d\, e^{-(\theta+2\lambda)\tau}}{-(\theta+2\lambda)}$$

$$= \frac{(2\lambda)^2}{-(\theta+2\lambda)} \int_0^\infty \tau\, d\, e^{-(\theta+2\lambda)\tau}$$

By $\int_0^\infty u\, dv = uv\Big|_0^\infty - \int_0^\infty v\, du$, where $u=\tau$, $v = e^{-(\theta+2\lambda)\tau}$

we have:

$$\int_0^\infty \tau\, d e^{-(\theta+2\lambda)\tau} = \tau\cdot e^{-(\theta+2\lambda)\tau}\Big|_0^\infty - \int_0^\infty e^{-(\theta+2\lambda)\tau}\, d\tau$$

$$= 0 - \frac{\int_0^\infty e^{-(\theta+2\lambda)\tau}\, d[-(\theta+2\lambda)\tau]}{-(\theta+2\lambda)}$$

$$= \frac{1}{\theta+2\lambda}\, e^{-(\theta+2\lambda)\tau}\Big|_0^\infty$$

$$= -\frac{1}{\theta+2\lambda}$$

Hence:

$$A^*[\theta] = \frac{(2\lambda)^2}{-(\theta+2\lambda)} \times \frac{1}{-(\theta+2\lambda)} = \left[\frac{2\lambda}{\theta+2\lambda}\right]^2$$

Note: this result can be generalized to Erlang-k distribution.

$$A^*[\theta] = \left[\frac{k\lambda}{\theta+k\lambda}\right]^k \qquad 7.5$$

Step #2: by the equation $1-\pi_0 = A^*[\mu\pi_0]$, we have

$$1-\pi_0 = \left(\frac{2\lambda}{\mu\pi_0+2\lambda}\right)^2 \Rightarrow$$

$$(1-\pi_0)(\mu^2\pi_0^2+4\lambda\mu\pi_0+4\lambda^2) = 4\lambda^2$$

$$\mu^2\pi_0^2+4\lambda\mu\pi_0+\cancel{4\lambda^2}-\mu^2\pi_0^3-4\lambda\mu\pi_0^2-4\lambda^2\pi_0-\cancel{4\lambda^2}=0$$

Dividing by μ^2:

$$\pi_0^2+4\rho\pi_0-\pi_0^3-4\rho\pi_0^2-4\rho^2\pi_0=0$$

$$\pi_0\left[\pi_0+4\rho-\pi_0^2-4\rho\pi_0-4\rho^2\right]=0$$

$$\pi_0^2+(4\rho-1)\pi_0+4\rho^2-4\rho=0 \longrightarrow \begin{cases} ax^2+bx+c=0 \\ x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} \end{cases}$$

$$\pi_0 = \frac{1-4\rho+\sqrt{(4\rho-1)^2-4(4\rho^2-4\rho)}}{2} \qquad \because \pi_0>0$$

$$= -2\rho+0.5+\sqrt{16\rho^2-8\rho+1-16\rho^2+16\rho}\,/2$$

$$= -2\rho+0.5+\sqrt{8\rho+1}\,/2$$

$$= -2\rho+0.5+\sqrt{2\rho+0.25}$$

So, $\pi_0 = -2\rho+0.5+\sqrt{2\rho+0.25}$ 7.5

b) $\lambda=15/hr$ $\mu=20/hr$ $\Rightarrow \rho=\lambda/\mu=\frac{15}{20}=0.75 \Rightarrow \pi_0=-2\rho+0.5+\sqrt{2\rho+0.25}$
$=0.3229$

Because for GI/M/1 systems we have

$\pi_n = \pi_0(1-\pi_0)^n$

So: $\pi_4 = \pi_0(1-\pi_0)^4 = 0.06787$

c) $L = \frac{\rho}{\pi_0} = \frac{0.75}{0.3229} = 2.3227$

d) $W_q = \frac{(1-\pi_0)W_s}{\pi_0} = \frac{(1-0.3229)*\frac{1}{20}}{0.3229}$

$= 0.104847$ hour

2. Lili Malign, an analyst at Luigi's Contract Service, is considering the queue discipline to use for one of the main office systems. Lili modeled this system as an M/D/1 queueing system with λ=5 customers per hour and Ws=9 minutes.
   Help her calculate L, W, L_q, W_q assuming
   a) FCFS (First-come, first-served), and
   b) LCFS (Last-come, first-served) non-preemptive queue discipline.

a) FCFS: $\lambda = 5/\text{hour} = \frac{5}{60}/\text{min} = \frac{1}{12}/\text{min}$

$W_s = 9$ min

10 (2.5) each

$\therefore \rho = \lambda \cdot W_s = \frac{45}{60} = \frac{3}{4}$

Based on the results of M/D/1 discussed in class:
(slide #2 in Queueing Systems (IV))

$L = \frac{\rho(2-\rho)}{2(1-\rho)} = \frac{\frac{3}{4}\times\frac{5}{4}}{2\times\frac{1}{4}} = \frac{15}{8}$

$W = L/\lambda = \frac{15}{8} \Big/ \frac{1}{12} = \frac{45}{2}$ min

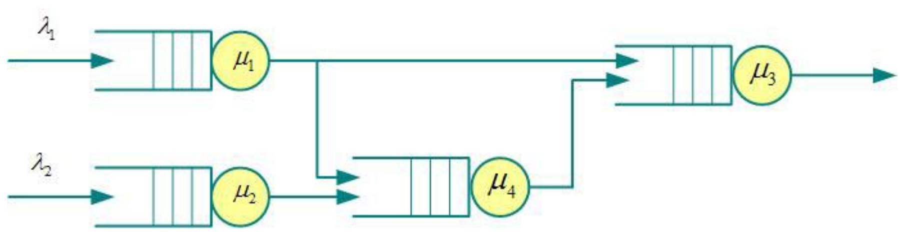
$W_q = W - W_s = \frac{45}{2} - 9 = \frac{27}{2}$ min

$L_q = W_q \times \lambda = \frac{27}{2} \times \frac{1}{12} = \frac{27}{24}$

b) By Theorem 5.4.1 (part of it discussed in class): 5

The performance measures L, W, L_q, W_q for LCFS are the same as for the FCFS discipline because LCFS discipline does not consider customer service times or any measure of them.

3. **Jackson Theorem/Algorithm:** Consider the following open queueing network:



Given that

- Number of nodes K=4
- Service times are exponentially distributed with
  $$\frac{1}{\mu_1} = 0.1, \quad \frac{1}{\mu_2} = 0.4, \quad \frac{1}{\mu_3} = 0.1, \quad \frac{1}{\mu_4} = 0.15$$
- The arrival to node 1 has Poisson pattern with rate $\lambda_1 = 5\, jobs/\sec$
- The inter-arrival time to node 2 is exponentially distributed with parameter $\lambda_2 = 2\, jobs/\sec$
- FCFS scheduling discipline is used for all nodes
- Transition probabilities are
  $p_{13} = 0.4$, $p_{14} = 0.6$, $p_{24} = 1$, $p_{43} = 1$

Answer the following questions:

a) Find the average arrival rates to each node?

b) Find the following performance measures:

   (1) Server utilization for each node

   (2) Mean response time for each node

   (3) Mean number of jobs at each node

   (4) Mean overall response time of the system

   (5) Marginal steady-state probabilities $\pi_1(1)$, $\pi_2(2)$, $\pi_3(3)$, $\pi_4(4)$

c) Find the joint steady-state probability $\pi(1,2,3,4)$?

(1) Based on the traffic equations $\Lambda_k = \lambda_k + \sum_{j=1}^{K} \Lambda_j P_{jk}$, we have

$$\left.\begin{aligned} \Lambda_1 &= \lambda_1 = 5 \\ \Lambda_2 &= \lambda_2 = 2 \\ \Lambda_3 &= \Lambda_1 \cdot P_{13} + \Lambda_4 \cdot P_{43} = 0.4\times5 + \Lambda_4 \\ \Lambda_4 &= \Lambda_1 \cdot P_{14} + \Lambda_2 \cdot P_{24} = 5\times0.6 + 2 = 5 \end{aligned}\right\} \Rightarrow \begin{aligned} \Lambda_1 &= 5 \\ \Lambda_2 &= 2 \\ \Lambda_3 &= 7 \\ \Lambda_4 &= 5 \end{aligned}$$

(10) points

(2) (a) server utilization.

$$\rho_1 = \frac{\Lambda_1}{\mu_1} = 5\times0.1 = 0.5 \qquad \rho_2 = \frac{\Lambda_2}{\mu_2} = 2\times0.4 = 0.8$$

$$\rho_3 = \frac{\Lambda_3}{\mu_3} = 7\times0.1 = 0.7 \qquad \rho_4 = \frac{\Lambda_4}{\mu_4} = 5\times0.15 = 0.75$$

(8)

For $i=1,2,3,4$, the stability condition $\rho_i < 1$ is satisfied!

(b) Mean response time.

$$\overline{T_1} = \frac{1/\mu_1}{1-\rho_1} = \frac{0.1}{1-0.5} = \frac{1}{5} = 0.2$$

$$\overline{T_2} = \frac{1/\mu_2}{1-\rho_2} = \frac{0.4}{1-0.8} = 2$$

$$\overline{T_3} = \frac{1/\mu_3}{1-\rho_3} = \frac{0.1}{1-0.7} = \frac{1}{3}.$$

$$\overline{T_4} = \frac{1/\mu_4}{1-\rho_4} = \frac{0.15}{1-0.75} = 0.6$$

(8)

(c) Mean number of jobs at each node:

$$\overline{k_1} = \frac{\rho_1}{1-\rho_1} = \frac{0.5}{1-0.5} = 1 \qquad \overline{k_2} = \frac{\rho_2}{1-\rho_2} = \frac{0.8}{1-0.8} = 4$$

$$\overline{k_3} = \frac{\rho_3}{1-\rho_3} = \frac{0.7}{1-0.7} = \frac{7}{3} \qquad \overline{k_4} = \frac{\rho_4}{1-\rho_4} = \frac{0.75}{1-0.75} = 3$$

(8)

(d). Mean overall response time of the system:

Based on the "superposition property" of Poisson process.

the overall arrival rate of the system can be calculated by:

$$\lambda = \lambda_1 + \lambda_2 = 7 \text{ jobs/sec}$$

(8)

The average (total) number of jobs in the system will be:

$$L = \overline{K} = \sum_{i=1}^{4} \overline{k_i} = 1+4+\frac{7}{3}+3 = \frac{31}{3}.$$

By Little's Law:

$$W = \frac{\overline{K}}{\lambda} = \frac{L}{\lambda} = \frac{\frac{31}{3}}{7} = \frac{31}{21} = 1.476$$

e. Marginal state probabilities,

$\pi_1(1) = (1-\rho_1)\rho_1 = 0.5 \times 0.5 = 0.25$

$\pi_2(2) = (1-\rho_2)\cdot\rho_2^2 = 0.2 \times 0.8^2 = 0.128$

$\pi_3(3) = (1-\rho_3)\cdot\rho_3^3 = 0.3 \times 0.7^3 = 0.1029$

$\pi_4(4) = (1-\rho_4)\rho_4^4 = 0.25 \times 0.75^4 = 0.079102$

8

3) Joint steady-state prob:

5 points

$\pi(1,2,3,4) = \pi_1(1) * \pi_2(2) * \pi_3(3) * \pi_4(4)$

$= 0.25 \times 0.128 \times 0.1029 \times 0.079102$

$= 0.00026$